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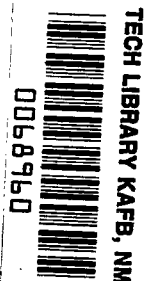
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DYNAMICS OF THE SPATIAL MOTION OF AN AIRCRAFT

by G. S. Byushgens and R. V. Studnev

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NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

Systems of instability in maneuvering aircraft, which do in fact appear with increase in speed and height of flight, have served in recent years as the basis for carrying out theoretical and experimental studies both in the Soviet Union and abroad. As a result, cross couplings between parameters have been established, characterizing the longitudinal and lateral motion of an aircraft.

On the basis of a number of their own investigations and a collection of papers published by other writers, the authors of this monograph discuss original concepts here of the theory of spatial motion of an aircraft by taking into account these cross couplings.

Criteria are given here for stabilizing the motion of an aircraft during maneuvers involving intense rolling. To describe the characteristics of aircraft dynamics during spatial maneuvers, the authors employ several concepts and methods involving the qualitative theory of differential equations. The qualitative results obtained in the analysis are illustrated by the results of calculating the aircraft dynamics on digital computers and simulators.

Considerable attention is paid in this book to a physical explanation of the results obtained and to a description of the mechanics of aircraft motion in the most common cases of controlled flight. Detailed analysis is given of the reasons and conditions for appearance of the phenomenon of "inertial rotation" of an aircraft involving losses in effective lateral control.

This book is intended for industrial engineers, instructors, and students in aviation institutes of technology. Included herein are 8 tables, 141 illustrations, and a bibliography of 76 names.

TABLE OF CONTENTS

FOREWORD

The design features of modern aircraft have caused the appear-
ance of a new type of motion instability during aircraft maneuvers
involving intense rolling. Simply analyzing the stability of
straight and steady flight for modern maneuvering aircraft, using
linear equations of motion, is not sufficient to select their basic
parameters.

A method for analyzing the stability of aircraft motion, in-
volving rotation relative to the longitudinal axis (rolling), was
developed by the authors in 1957-1964, independent of the work done
by Phillips in 1948, who was unknown to the authors, although
numerous citations are found in the foreign literature.

This monograph is devoted to analyzing the above class of air-
craft spatial motions.

The stability and controllability of spatial motions of an air-
craft at a constant flying speed are studied in this book, i.e.,
the so-called "rapid" motions, associated mainly with angular rolling
velocity. In general form, the problem is rather complicated, since
nonlinear equations of motion are analyzed and an attempt is made
here to simplify them as much as possible. In a number of cases,
methods are employed for analyzing steady motion. For the more
complex problems, for example the evaluation of transient conditions
during control, the results are cited from solutions obtained on
computers.

Here also the motion of an aircraft is studied as a function
of aileron, elevator and rudder deflection and physical explanations
are given for the results obtained. Several concepts and methods
employing the qualitative theory of differential equations are
widely used in the analysis to describe the properties of the spatial
motion of an aircraft and to achieve the most common results.

Considerable attention is given here to studying the possibility
of stability and controllability loss by an aircraft during intense
rolling (inertial rotation system). Conditions are cited under
which it is possible for an aircraft to enter into such systems and
physical explanations are given for such phenomena.

The authors wish to express their appreciation to the reviewer

of this book, Prof. V.N. Matveyev, who contributed a number of useful comments in looking over the manuscript, and to Engineer M.M. Medvedyev who assisted in the computer calculations and in the simulation.

Since this monograph is a first attempt at illuminating the rather complex phenomenon of the dynamics of modern supersonic aircraft, the authors would be most appreciative of any comments the readers would like to make. The address is: Moscow, K-51, Petrovka, 24, "Mashinostroyeniye" Press.

INTRODUCTION

The basic method used in the "classical" theory of aircraft
motion relative to the center of mass is that of linearizing the
equations of motion. In the majority of cases this method permits
a system of equations to be simplified by dividing it into two inde-
pendent systems of equations of longitudinal and lateral motion.
A large number of books have been devoted to describing the results
obtained through such simplifications (see e.g., [18]-[36]), in
which the simplest forms of aircraft motion are analyzed when the
deviations of all parameters from the nominal values are small. It
is natural that the fundamental questions underlying the analysis
here are those involving the stability of motion "in a small region"
and, in certain instances, the study of transient conditions acted
on by weak disturbances and small deflections of the control surfaces.

In nonlinear formulation, only the plane motion of an aircraft
in a longitudinal plane has been studied (see, e.g., [33]). For a
long time the dynamic characteristics of aircraft could have been
evaluated satisfactorily by analyzing such simplified models of
motion. However, the increase in speed and height of flight, which
has been the reason for substantial changes in the geometric and
inertial characteristics, has caused the characteristics of aircraft
stability and controllability, to be linearly dependent on the
parameters of its motion, especially during maneuvers involving
rolling. In particular, systems of instability have been discovered,
theoretically and experimentally, which could not be defined by
simplified analysis when all parameters of motion were assumed
small. Such dynamic characteristics of maneuvering aircraft are
associated with the existence of cross couplings between parameters,
which are characteristic of longitudinal and lateral motion, thus
making it impossible to separate the equations and requiring math-
ematical expression in the analysis of nonlinear differential equa-
tions. V.S. Pyshnov [25] apparently was the first to give any
attention to cross couplings between longitudinal and lateral motion.
However, he studied only the slight interactions caused by the aero-
dynamic cross couplings, which exert an insignificant influence on
the motion of an aircraft relative to the center of mass.

Theoretically the new effects were discovered with respect to
the influence of so-called inertial cross coupling on the motion of
an aircraft, especially during maneuvers at high rolling velocities.
According to the foreign data, the first theoretical work in this

direction was apparently that of Phillips.*

After several aviation catastrophes involving American crashes**
the causes of which were sought in the manifestation of inertial
cross couplings, interest grew in this problem, thus resulting in
a large amount of pertinent research (see [37]-[59]), concerned
mainly with stability conditions in the steady rotation of an air-
craft at high angular rolling velocities, and also analysis of
certain dynamic characteristics found by calculations performed on
simulators.

Quite detailed results are given in the work by Pinsker [50]
on determining values of the maximal deflections by angles of attack
and side slipping, obtained in studying the angle of turn at a
given angle of bank. All these results have been published in the
form of individual journal articles and individual issues of supple-
ments, therefore they are inconvenient to use and require systemizing
and unifying.

In spite of the comparatively long developmental history of the
theory of aircraft motion relative to the center of mass there is
still no sufficiently complete presentation of the dynamic char-
acteristics of an aircraft during maneuvers with large variations
in the parameters of motion. Research, which has been conducted in
studying inertial cross couplings, has created a base on which it
is possible to construct a general theory of aircraft motion rela-
tive to the center of mass. To some degree such an attempt is being
made in this book.

First let us say that the problem of investigating the dynamics
of an aircraft in the most general formulation, when we look simul-
taneously at the motion of an aircraft both relative to the center
of mass and the motion of its center of mass, is quite complex and
is not analyzed in this paper. In our book we have made an attempt
to solve the more modest problem of investigating the qualitative
characteristics of aircraft motion relative to the center of mass
on the assumption that, for the time of such motion, the motion
characteristics of its center of mass do not vary, i.e., its speed
and height of flight are assumed to be constant. In addition to
simplifying the computations, all fundamental investigations are
conducted on the assumption that the aerodynamic coefficients of an
aircraft are linear functions of their own arguments. It should be
noted that these two assumptions, generally speaking, are not

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*W.H. Phillips, "Effect of Steady Rolling on Longitudinal and
Directional Stability", NACA TN, June, 1948.

**The foreign press noted that, as a result of the unfavorable ef-
fect of inertial cross couplings, several North American Super Sabre
F-100 and an experimental plane (the Bell X-2) were destroyed in
the air.

excessively restrictive, since the basic purpose of the work con-
sists in determining the qualitative characteristics of spatial
controlled motion of an aircraft under ordinary (rather than spin)
flying conditions. With such limitations, the variability in flying
conditions and the nonlinearity of the aerodynamic coefficients
lead mainly to a slight quantitative change, however, the qualitative
characteristics of the motion are retained. All the results obtained
in the research, as well as the methods, can be extrapolated to the
case of nonlinear aerodynamics.

Since the major attention in this research is devoted to the
discovery of qualitative characteristics of motion, all the numeri-
cal results are merely illustrative.

At the present time, due to the widespread use of simulators
and especially digital computers, the problems involved in finding
precise solutions have been substantially simplified. In this case
the significance of the methods, which permit a qualitative picture
to be found of the motion (even though admittedly approximate),
increases since such methods substantially simplify the solution to
problems by allowing the qualitative character of the results to be
predicted and by reducing the amount of machine time necessary for
analyzing the motion of an aircraft. In this respect we may note
that, at the present time, there is apparently no value in carrying
out detailed approximate parametric computations for any specific
cases of motion, since the system of equations of aircraft motion
is a multiparametric one and requires a large amount of computation
to develop standard cases. The methods of mathematical modeling
yield much greater potentiality in this respect.

The problems involved in studying the motion of symmetric
rotating missiles are very similar. The problems involved in in-
vestigating the stability of motion of rotating axisymmetric artil-
lery shells and jet-propelled finned missiles in linearized formu-
lation, have been studied by any number of authors (see [60]-[62]),
however the major new phenomena in the dynamics of missiles have
been found rather recently (see [63]-[76]). These phenomena are
related to the so-called "resonance" of lateral oscillations of a
missile with an angular rolling velocity and arise when unbalanced
moments, either aerodynamic or from thrust misalignment, are present.
It is interesting to note that such resonance phenomena, in several
cases leading to a real loss in stability of the missile, may ap-
pear when the rotating finned missiles are in motion and are not
observed in the dynamics of artillery shells. This effect is due
to the phenomenon of resonance being possible only for statically
stable missiles and impossible for statically unstable missiles,
in particular, for artillery shells, which are usually aerodynamical-
ly unstable.

This book consists of seven chapters.

The first chapter is devoted to equations of spatial motion of
an aircraft; they are simplified relative to the above-formulated

problems and are reduced to dimensionless form.

The second chapter involves a study of the stability of motion of an aircraft during steady rotation relative to the longitudinal axis. In this chapter we cite criteria of stability and analyze the boundaries of the stability regions as a function of the different parameters. Such a study of motion is essential in analyzing the more general cases of aircraft motion involving aileron control.

The third and subsequent chapters are an analysis of the spatial motion of an aircraft using the methods of the qualitative theory of differential equations. The types of singular points are determined with various control surface deflection ratios; criteria are given for a periodic stability of motion in the vicinity of the singular points and the character of such motion is analyzed.

The last chapter cites several results of studying the dynamics of symmetric rotating missiles.

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CHAPTER I

EQUATIONS OF SPATIAL MOTION OF AN AIRCRAFT

1.   General Comments.   Equations of Motion for Solids.

In this book, as we noted in the Introduction, we look at sev-
eral classes of spatial motion of an aircraft and analyze its
stability and controllability. We have not studied the motion of
an aircraft along its flight path. At the present time there have
been any number of monographs and special educational books con-
cerned with the problems of analyzing the flight paths of an air-
craft (see [33], [34], [35], etc.). In studying the flight paths
we have devoted our greatest attention to solving differential
equations of motion, which follow from the theory of momentum, i.e.,
to solving the force equations, and have analyzed the equations of
moments, as a rule, only from the viewpoint of an approximate evalu-
ation of the required efficiency of controls or have not studied
them at all.

In correspondence with the basic problem formulated above, we
arrive at solutions to the equations of motion of an aircraft in a
form which is the most convenient for analyzing problems of stability
and controllability.

We shall assume that an aircraft is an absolutely rigid body
with a constant mass. Thus, we can assume that the liquid propel-
lant, which occupies considerable space in certain modern aircraft,
will be fixed in a position that corresponds to the original straight
and steady flight path. In special cases, when necessary, the in-
fluence of the motion of the liquid in the tanks on the stability
and dynamics of controlled motion must be analyzed specially.

We shall assume that the influence of structural elasticity
is expressed only in the values of the respective aerodynamic char-
acteristics. This influence can be taken into account quasistatical-
ly according to the mean values of the dynamic heads. Thus, we do
not take into account the increase in the number of degrees of free-
dom caused by structural elasticity. When necessary this can be

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*Numbers in the margin indicate pagination in the foreign text.

allowed for by methods which are specific for each given problem
(problems of studying the operation of automatic equipment, studying
flutter, etc.).
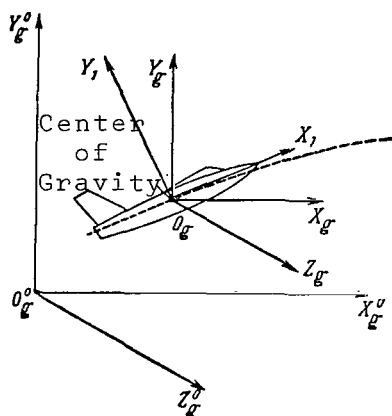
Fig. 1.1

To solve the equations of motion
we must define the coordinate system
in which the study will be conducted.

Let us look at the following coor-
dinate systems:

(a) As the reference coordinate
system, which is fixed relative to in-
ertial space, let us take the so-called
geodesic coordinate system $0_g^0 X_g^0 Y_g^0 Z_g^0$,
where the axis is oriented vertically
with the positive direction upward.
The axis $0_g^0 X_g^0$ can be directed horizon-
tally and its orientation can be chosen
arbitrarily; however it is usually
feasible to use the direction of flight
of the aircraft. The origin of this system can be set at a certain
given point on the earth's surface 0_g^0 (Fig. 1.1). In this coordinate
system it is convenient to measure the linear motion of an aircraft
relative to the Earth (i.e., the flight path of its center of mass);

(b) Let us select a central geodesic coordinate system $0_g X_g$
$Y_g Z_g$ such that its axes are parallel to the axes of the fixed refer-
ence geodesic system; the origin corresponds to the center of mass
of the aircraft (see Fig. 1.1) and moves in a forward direction
with it.

(c) Let us select the so-called system of fixed coordinate
axes, localized with the aircraft, and place its origin in the center
of mass of the aircraft. This coordinate system not only moves with
the aircraft but turns with it as well. We shall return to the
problem of orienting the axes of this sliding coordinate system
somewhat later.

The equations of motion of an aircraft, as a solid, can be ob-
tained from the laws of conservation of momentum and moment of
momentum. We can thus divide the equations into two groups, one of
which describes the motion of the center of mass of an aircraft and
the other - the motion about the center of mass. In vector form
these equations can be written as follows:

$$m \frac{d\overline{V}}{dt} = \overline{R} + \overline{G}$$ (1.1a)

and the equations of motion of an aircraft about the center of

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mass can be written as:

$$\frac{d\overline{K}}{dt} = \overline{M}. \qquad (1.1b)$$

In these equations we have used the following definitions:

V is the velocity vector of an aircraft about an inertial
space (axes $O_g^0 X_g^0 Y_g^0 Z_g^0$);

$\overline{R}$ is the vector of external forces acting on the aircraft;

$\overline{G}$ is the gravity vector;

$\overline{K}$ is the moment vector of momentum of an aircraft;

$\overline{M}$ is the moment vector of external forces about the cen-
ter of mass of an aircraft.

It is most convenient to conduct our analysis of the motion of
an aircraft relative to the center of mass in a sliding coordinate
system. Here the parameters of the motion of a solid, especially
the forward and angular speeds of motion, must be determined in a
sliding coordinate system. Thus, in carrying out the differentia-
tion in equations (1.1a) and (1.1b) we must use the formula for the
total derivative of the vector, given by vector analysis, that is
defined through the derivative relative to the sliding coordinate
system and the vector of the angular rotational velocity of the
sliding axes $\overline{\Omega}$, in the form

$$\left[\frac{d\overline{A}}{dt}\right] = \frac{d\overline{A}}{dt} + \overline{\Omega} \times \overline{A}, \qquad (1.2)$$

where [ ] is the symbol which denotes the total derivative;

$\overline{A}$ is the vector, that is defined in the sliding coordinate
system by projections a_x, a_y, a_z on the sliding axes $OXYZ$
and by the unit vectors i, j, k;

$\overline{\Omega}$ is the vector of the angular velocity of rotation of the
sliding system of coordinate axes relative to the fixed sys-
tem with projections ω_x, ω_y, ω_z on the sliding axes.

By using the vector produce $\overline{\Omega} \times \overline{A}$, we take into account that
component of change in the sliding coordinate system of the vector
derivative $\overline{A}$, which is caused by rotation of the sliding axes.
Formula (1.2) can otherwise be written in expanded form:

$$\left[\frac{d\overline{A}}{dt}\right] = \frac{d\overline{A}}{dt} + \begin{vmatrix} i & j & k \\ \omega_x & \omega_y & \omega_z \\ a_x & a_y & a_z \end{vmatrix}. \qquad (1.3)$$

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If we bear in mind the formula for differentiation (1.3), from the first vector equation (1.la) we find the system of equations for the motion of the center of mass of an aircraft in projections on the sliding axes:

$$m\left(\frac{dV_x}{dt}+\omega_y V_z-\omega_z V_y\right)=R_x+G_x;$$

$$m\left(\frac{dV_y}{dt}+\omega_z V_x-\omega_x V_z\right)=R_y+G_y; \qquad (1.4)$$

$$m\left(\frac{dV_z}{dt}+\omega_x V_y-\omega_y V_x\right)=R_z+G_z.$$

We call this group of equations the equations of force. We use the following definitions in them:

$OXYZ$ is the system of sliding coordinate axes, in which the motion of the aircraft is studied;

R_x, R_y, R_z are the projections of the external forces, acting on the aircraft;

G_x, G_y, G_z are the projections of gravity on the axes $OXYZ$.

The motion of a solid (an aircraft) relative to the center of mass is described by the second vector equation (1.lb).

From theoretical mechanics we know (see [9], [10]) that projections of the vector of the kinetic moment $\overline{K}$ on the sliding axes can be written in the general case in the following form:

$$K_x=J_x\omega_x-J_{xy}\omega_y-J_{xz}\omega_z,$$

$$K_y=-J_{xy}\omega_x+J_y\omega_y-J_{yz}\omega_z, \qquad (1.5)$$

$$K_z=-J_{xz}\omega_x-J_{yz}\omega_y+J_z\omega_z.$$

The first simplification of equations (1.5) is due to the existence of a pitching plane in practically all types of aircraft. If we select the axes OXY such that they lie in the pitching plane of the aircraft, we find that $J_{xz} = J_{yz} = 0$. From expression (1.lb), if we use equations (1.5) and carry out the differentiation by taking (1.3) into account, we find the second system of equations for the motion of a solid, having a pitching plane:

$$J_x\frac{d\omega_x}{dt}-J_{xy}\frac{d\omega_y}{dt}+(J_z-J_y)\omega_y\omega_z+J_{xy}\omega_x\omega_z=M_x;$$

$$J_y\frac{d\omega_y}{dt}-J_{xy}\frac{d\omega_x}{dt}+(J_x-J_z)\omega_z\omega_x-J_{xy}\omega_y\omega_z=M_y; \qquad (1.6)$$

$$J_z\frac{d\omega_z}{dt}-J_{xy}\left(\omega_x^2-\omega_y^2\right)+(J_y-J_x)\omega_x\omega_y=M_z.$$

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We call this group of equations the equations of moments.

2. Equations of Motion for an Aircraft in Dimensional Form

In using the equations of motion, given above for a solid, to /13
describe the motion of an aircraft they must be supplemented by
specific expressions for the external forces and moments acting on
the aircraft as well as by expressions for the forces of gravity.
To obtain these supplementary data, we must in turn introduce a
definiteness into the orientation of the sliding system of coordinate
axes relative to the aircraft.
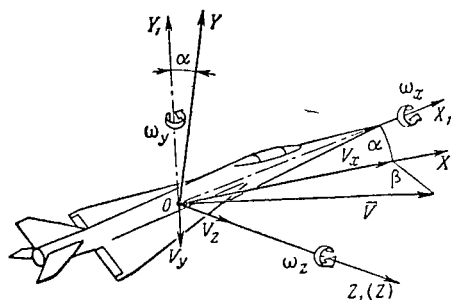
Fig. 1.2.

In studying the dynamics and aero-
dynamics of aircraft there are mainly
two coordinate systems which are ordi-
narily used: the so-called body-system
of coordinates and the semifixed sys-
tem. The semifixed coordinate system
$OXYZ$, where the axis OX is directed
along the velocity vector when $\beta = 0$,
has been used most widely in developing
and analyzing aerodynamic coefficients
for experiments in wind tunnels. Such
choice is due in large measure to the
peculiarities of measuring forces and
moments using aerodynamic balance, set
up in a given manner relative to the current of air in the wind
tunnel. In studying the dynamics of an aircraft in semifixed axes
we can use the aerodynamic coefficients obtained in such tests in
wind tunnels with no conversion. The arrangement of semifixed axes
and the main characteristic direction, selected on the aircraft in
the form of a mean aerodynamic chord or of a certain axis on the
fuselage, is illustrated in Figure 1.2.

However the system of semifixed coordinate axes has the same
disadvantage that the inertial moments of the aircraft, computed
relative to such axes, depend on the angle of attack and consequent-
ly are variable values, thus complicating the system of equations
of motion. These disadvantages are lacking in the body-system of
coordinates, fixed relative to the aircraft in which the inertial
moments are independent of the angle of attack.

The location of the axes $OX_1Y_1Z_1$ relative to the aircraft is
seen on Figure 1.2. In a body-system of coordinates the equations
of motion appear most simply, if the major inertial axes of the air-
craft are taken for the axes $OX_1Y_1Z_1$. For the majority of dynamic /14
problems the body-system of coordinate axes is physically the most
justified since the measuring instruments, the control gauges and
even the pilot himself are in a body-system of coordinates and re-
act to its motion. We make our studies in this book, on the basis
of the above discussions, in a body-system of central coordinate
axes, whereby we take the major axes of inertia. However in this

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case it is essential that the aerodynamic forces and moments obtained in the wind tunnels be converted to other axes.

The aerodynamic forces and moments which are on the right-hand side of the equations of motion depend on height and speed of flight, angles of attack and side slip, angular velocities of rotation of the aircraft and are independent of its orientation relative to the Earth. Only the projections of gravity (G_x, G_y, G_z) on the axes of the sliding coordinate system depend on the orientation of the craft relative to the Earth. To locate these projections it is essential to introduce angles which define the orientation of the aircraft relative to the Earth. The orientation of the fixed axes relative to the geodesic coordinate axes $OX_g Y_g Z_g$ are usually determined by using the Euler angles ϑ, ψ, γ, whose meaning is obvious from the method of sequential turns of the fixed axes relative to the axes $OX_g Y_g Z_g$ used to find them.

Fig. 1.3.

The **course angle** ψ (angle of yaw) is defined as the angle between a given reference direction (axis OX_g) and the projections of the body axis OX_1 on the horizontal plane.

The angle between the body axis OX_1 and the horizontal plane is termed the **angle of pitch** and is denoted by ϑ.

As the **angle of bank** we understand the angle between the vertical plane, passing through the axis OX_1, and the body axis OY_1 of the aircraft.

Figure 1.3 shows a system of axes $OX_g Y_g Z_g$ that is fixed relative to the Earth and a system of axes $OX_1 Y_1 Z_1$ that is fixed with the aircraft. We shall assume that the systems of coordinate axes are superimposed at the origin. We show how by sequential turns of the body axes we can obtain the Euler angles introduced above (see Fig. 1.3). The first turn of the system of body axes can be made relative to the axis OY_g for the heading ψ ($\dot{\psi}$ corresponds with the axis OY_g); the second can be made relative to the axis OZ' at an angle ϑ ($\dot{\vartheta}$ corresponds with the axis OZ') and finally the third turn can be made relative to the axis OX_1 at an angle γ ($\dot{\gamma}$ corresponds with the axis OX_1). By projecting the vectors $\dot{\psi}, \dot{\vartheta}, \dot{\gamma}$, which appear as the components of the vector of angular velocity of the motion of the aircraft relative to the fixed coordinate system for the body axes of the craft, we find the coupling equations between the Euler angles and the angular velocities of the body axes:

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$$\begin{aligned}
\omega_x &= \dot{\gamma} + \dot{\psi}\sin\vartheta; \\
\omega_y &= \dot{\psi}\cos\vartheta\cos\gamma + \dot{\vartheta}\sin\gamma; \\
\omega_z &= \dot{\vartheta}\cos\gamma - \dot{\psi}\cos\vartheta\cdot\sin\gamma.
\end{aligned} \qquad (1.7)$$

By solving these equations relative to the derivatives $\dot{\gamma}$, $\dot{\psi}$ and $\dot{\vartheta}$, we find the kinematic relationships which couple the derivatives of the Euler angles and the projections of the vector of angular velocity of the aircraft on the body axes

$$\begin{aligned}
\dot{\vartheta} &= \omega_y\sin\gamma + \omega_z\cos\gamma; \\
\dot{\psi} &= \frac{1}{\cos\vartheta}(\omega_y\cos\gamma - \omega_z\sin\gamma); \\
\dot{\gamma} &= \omega_x - \text{tg}\,\vartheta\,(\omega_y\cos\gamma - \omega_z\sin\gamma).
\end{aligned} \qquad (1.8)$$

Using the Euler angles, and the expressions given in Table 1 for the direction cosines, it is easy to find expressions for the projections of gravity on the body axes

$$\begin{aligned}
G_x &= -G\sin\vartheta; \\
G_y &= -G\cos\vartheta\cos\gamma; \\
G_z &= G\cos\vartheta\sin\gamma.
\end{aligned} \qquad (1.9)$$

TABLE 1

Geodesic Axis / Body Axis	OX_g	OY_g	OZ_g
OX_1	$-\cos\psi\cos\vartheta$	$\sin\vartheta$	$-\sin\psi\cos\vartheta$
OY_1	$-\cos\psi\sin\vartheta\cos\gamma + \sin\psi\cdot\sin\gamma$	$\cos\vartheta\cos\gamma$	$\cos\psi\sin\gamma + \sin\psi\sin\vartheta\cos\gamma$
OZ_1	$\cos\psi\sin\vartheta\sin\gamma + \sin\psi\cos\gamma$	$-\cos\vartheta\sin\gamma$	$\cos\psi\cos\gamma - \sin\psi\sin\vartheta\sin\gamma$

It then follows to note that equations (1.8) have singularity at the angle of pitch $\vartheta \to \pi/2$, since in such case the values $\vartheta\cos\vartheta \to 0$. In those cases when it is necessary to study the motion at

$\vartheta \rightarrow \pi/2$, we must take this singularity into account and determine the Euler angles in another manner, or simply consider the equations without singularities. Such equations, in particular, are found in the work by Shilov [17].

In order that the equations of motion (1.4), (1.6) be integrated, we must add to them the equation of change in altitude and air density in the process of aircraft motion:

$$\frac{dH}{dt} = V_x \sin \vartheta + V_y \cos \vartheta \cos \gamma - V_z \cos \vartheta \sin \gamma; \left.\begin{array}{c} \\ \\ \end{array}\right\} \qquad (1.10)$$
$$\varrho = \varrho_0 e^{-\lambda H},$$

where λ is a constant equal to $\cong \frac{1}{1000} \left[\frac{1}{m}\right]$;

ρ_0 is the density of air at $H = 0$.

We must also take into account the relationship between the projections of flying speed on the body axes, the angle of attack α and the angle of side slip β (see Fig. 1.2):

$$V_x = V \cos \alpha \cos \beta; \left.\begin{array}{c} \\ \\ \\ \end{array}\right\}$$
$$V_y = - V \sin \alpha \cos \beta; \qquad (1.11)$$
$$V_z = V \sin \beta,$$

where V is the total flying speed.

Relationships (1.11) can be assumed as a definition for the angles of attack and side slip, from which it follows that by the angle of attack α we mean the angle between the projections of the velocity vector $\overline{V}$ of the aircraft on its pitching plane and its axis OX_1 ($\alpha > 0$, when the axis OX_1 is located above the projections of the velocity vector, see Fig. 1.2). The angle of side slip β is defined as the angle between the velocity vector and the pitching plane OX_1Y_1 of the aircraft ($\beta > 0$ if the aircraft flies with the right wing "forward").

Below we shall analyze only the controlled motion of an aircraft, therefore we assume that there are no disturbances in the atmosphere. Consequently, the flying speed and the up-stream velocity are reciprocally equal and differ in direction by an angle of 180°. /17

Let us introduce the aerodynamic forces and moments through their dimensionless coefficients[*] and once more rewrite the force

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[*]In future sections, the index "1" in the dimensionless coefficients will be omitted for brevity.

and moment equations:

$$m\left(\frac{dV_x}{dt} + \omega_y V_z - \omega_z V_y\right) = -c_{x1}\frac{\varrho V^2}{2}S - G\sin\vartheta + P_{eng}\cos\phi$$

$$m\left(\frac{dV_y}{dt} + \omega_z V_x - \omega_x V_z\right) = c_{y1}\frac{\varrho V^2}{2}S - G\cos\vartheta\cos\gamma +$$

$$+ P_{eng}\sin\phi \qquad (1.12)$$

$$m\left(\frac{dV_z}{dt} + \omega_x V_y - \omega_y V_x\right) = c_{z1}\frac{\varrho V^2}{2}S + G\cos\vartheta\sin\gamma,$$

where P_{eng} is the engine thrust, comprising the angle ϕ with the axis OX_1 (we now assume $\phi = 0$).

The moment equations are:

$$J_x\frac{d\omega_x}{dt} + (J_z - J_y)\omega_y\omega_z = m_{x1}\frac{\varrho V^2}{2}Sl,$$

$$J_y\frac{d\omega_y}{dt} + (J_x - J_z)\omega_z\omega_x = m_{y1}\frac{\varrho V^2}{2}Sl, \qquad (1.13)$$

$$J_z\frac{d\omega_z}{dt} + (J_y - J_x)\omega_x\omega_y = m_{z1}\frac{\varrho V^2}{2}Sb_A.$$

As we know, the coefficients of aerodynamic forces and moments in the general case are expressed in the form of functional dependences of the kinematic parameters of motion and the parameters which define the flight path

$$c_{x,y,z} \text{ or } m_{x,y,z} = F(\alpha, \beta, \delta_a, \delta_e, \delta_r, \omega_x, \omega_y, \omega_z, M, \text{Re}). \qquad (1.14)$$

In relationships (1.14) we use the definitions: M = Mach number; Re = $V_0 b_A / \nu$ = the Reynolds number.

This last group of parameters (M,Re) characterizes the initial flight path, therefore in analyzing the stability or controllable motions, these parameters may be taken as constant values. In analyzing the stability of motion with fixed controls it follows also to take as constants the angles of deflection of the control /18 surfaces δ_a, δ_e, δ_r. The effect of the angular velocities on the force coefficients, though some does exist, is not large and has been insufficiently studied. Thus, we can write the following basic functions, which are advisable to take into account in studying the motion of an aircraft relative to the center of mass:

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$$c_{x1} = f_1 (\alpha, \beta, M, \delta_a, \delta_e, \delta_r);$$
$$c_{y1} = f_2 (\alpha, \beta, M, \omega_z, \delta_e);$$
$$c_{z1} = f_3 (\beta, M, \omega_x, \omega_y, \delta_a, \delta_r);$$
$$m_{x1} = f_4 (\alpha, \beta, M, \omega_x, \omega_y, \delta_a, \delta_r);$$
$$m_{y1} = f_5 (\alpha, \beta, M, \omega_x, \omega_y, \delta_a, \delta_r);$$
$$m_{z1} = f_6 (\alpha, \beta, M, \omega_z, \delta_e).$$

$$(1.15)$$

Consequently, in the general case of motion, on the right-hand side of each of the equations of forces and moments a rather complex function is contained which as a rule can be determined on the basis of approximations of experimental data. The total system of equations of spatial motion of an aircraft obtained is quite complex and in general form can be solved only on computers. However to obtain answers to the whole series of questions it is necessary only to solve the simplified equations. Examples of such simplifications are the division of aircraft motion into longitudinal and lateral motion and the further subdivision of the longitudinal motion into short-range and long-range. Such an approach to analyzing the dynamics of an aircraft is possible when the deviations from the original flight path are small and the equations can be linearized.

3. Linearization of the Equations of Motion for an Aircraft. Conditions for Dividing the Total System of Equations into Independent Systems.

In analyzing any engineering problem, properly chosen assumptions which take into account the most essential factors, permit obtaining a system of differential equations of motion that is more simple than the original one.

Let us linearize the equations of motion of an aircraft given above. Then let there exist a certain solution to the system of equations obtained above. We shall assume that this solution will correspond to a certain steady flight path

$$V_0; \ \alpha_0; \ \gamma_0 = \omega_{x0} = \omega_{y0} = \omega_{z0} = 0.$$

The motion or the state of equilibrium, corresponding to this solution will be termed undisturbed motion. According to the usually accepted method of linearization we shall assume that the disturbed motion is determined by small increases in the basic parameters toward their undisturbed values, i.e., $V = V_0 + \Delta V$, $\alpha = \alpha_0 + \Delta\alpha, \ldots$, etc. The reason for the disturbed motion of an aircraft may be small changes in the initial conditions, small deflections in the controls or any other small disturbances.

Let us expand the functions in the right-hand sides of the equations of systems (1.12) and (1.13) into a series for the basic kinematic parameters of the motion and retain only those terms which contain first-order derivatives.

The condition of smallness of the deviations permits assuming that the sines of the increase for all angles are equal to the angle itself and the cosines are equal to unity. As a result of the smallness of the analyzed reference angles of attack and side slip this assumption may be approximately expanded to their original values. From the above, in particular, it follows that

$$V_x = V; \; V_y = -V\alpha; \; V_z = V\beta. \qquad (1.16)$$

In linearizing the forces and moments acting on the aircraft, we must take into account that the linearization is made at the point of equilibrium, which in this case corresponds to the straight and steady flight with no side slip. From the existence of a pitching plane of the aircraft X_1OY_1, it follows that all derivatives of the longitudinal moment and forces acting in this plane, according to the kinematic parameters which determine the nonsymmetric part of the motion (β, ω_x, ω_y), will be equal to zero.

Analogously, all the derivatives of the "asymmetric" forces and moments according to the parameters which characterize the motion in the pitching plane of the aircraft, will also be identically equal to zero in the range below the critical angles of attack, i.e., until the flow occurs without separation.

If we drop the terms above the first order of smallness and bear in mind the equation of equilibrium of forces and moments for the straight and steady flight, we find the familiar equations of motion of an aircraft in the variations:

$$
\left.
\begin{aligned}
m\,\frac{dV_x}{dt} &= -G\cos\vartheta_0\,\vartheta - X^\alpha\alpha - X^V V; \\
m\left(\frac{dV_y}{dt} + V_0\omega_z\right) &= G\cdot\sin\vartheta_0\,\vartheta + Y^\alpha\alpha + Y^V V; \\
J_z\,\frac{d\omega_z}{dt} &= M_z^\alpha\alpha + M_z^V V + M_z^{\omega_z}\omega_z + M_z^{\dot\alpha}\dot\alpha; \\
\frac{d\vartheta}{dt} &= \omega_z;
\end{aligned}
\right\}
\qquad (1.17a)
$$

$$m \left(\frac{dV_z}{dt} - V_0 \omega_y \right) = G \cos \vartheta_0 \gamma + Z^\beta \beta;$$

$$J_x \frac{d\omega_x}{dt} = M_x^{\omega_x} \omega_x + M_x^{\omega_y} \omega_y + M_x^\beta \beta;$$

$$J_y \frac{d\omega_y}{dt} = M_y^{\omega_x} \omega_x + M_y^{\omega_y} \omega_y + M_y^\beta \beta;$$

$$\frac{d\gamma}{dt} = \omega_x - \tan\vartheta_0 \cdot \omega_y.$$

(1.17b)

In these equations for brevity we omit the sign of the incre-
ment Δ, and the parameters of the original undisturbed motion are
denoted by the index "zero".

As we can see, the system of equations given here can be
divided into two systems which are independent of one another (1.17a)
and (1.17b). One of them, system (1.17a), describes the change in
parameters of the longitudinal motion of an aircraft V_y, V, ω_z;
the other system, (1.17b), describes the change in parameters of
the lateral motion ω_x, ω_y, γ and V_z.

Consequently, the so-called longitudinal disturbed motion
in the vertical plane can be distinguished from the lateral dis-
turbed motion (rolling and yawing at a constant flying speed V_0 and
a constant angle of attack α_0). Each of these motions is described
by a system of linear differential equations of fourth order; to
find the solutions to these equations we must find the roots of the
algebraic equation of fourth order and determine the independent
constants for the initial conditions. The analytical methods for
a disturbed motion of an aircraft in this formulation have been
developed quite well and in detail (see, for example, the book by
V.N. Matveyev [31]).

In the same instance when it is necessary to look at the
controlled flight of an aircraft which may be accompanied by the
development of high angular velocities, we must retain in the equa-
tions of motion the nonlinear terms which contain the angular
velocities. Let us write the equations for small α and β.

We introduce first the supplemental relationship for the
angle of attack:

$$\alpha = \alpha_m + \alpha_1,$$

(1.18)

where α_m is the angle between the axis of the aircraft, correspond-
ing to zero lift, and the major inertial axis ($\alpha_m > 0$, when OX_1 lies
above the axis of zero lift). In such definitions the angle of
attack α_1 is measured from the axis which corresponds to zero lift[*].

───────────
[*]The index "1" will now be omitted.

If we substitute relationship (1.18) into equations of motion (1.17a) and (1.17b) and carry out the necessary transformations, we find /21

$$
\left.
\begin{aligned}
&-\dot{\alpha}+\omega_z-\beta\omega_x-\frac{\dot{V}}{V}(\alpha_m+\alpha)=\overline{Y}^\alpha\cdot\alpha+\overline{Y}^\delta e\,\delta_e-\frac{g}{V}\cos\vartheta\cos\gamma; \\
&\dot{\omega}_z+A\omega_x\omega_y=\overline{M}_z^\alpha\,\alpha+\overline{M}_z^{\bar{\omega}}z\cdot\omega_z+\overline{M}_z^{\dot{\alpha}}\cdot\dot{\alpha}+\overline{M}_z^\delta e\cdot\delta_e; \\
&\dot{\beta}-\omega_y+\frac{\dot{V}}{V}\beta-\omega_x(\alpha_m+\alpha)=\overline{Z}^\beta\cdot\beta+\overline{Z}^{\delta_r}\delta_r+\frac{g}{V}\cos\vartheta\sin\gamma; \\
&\dot{\omega}_y-B\omega_x\omega_z=\overline{M}_y^\beta\cdot\beta+\overline{M}_y^{\bar{\omega}}y\omega_y+\overline{M}_y^{\bar{\omega}}x\cdot\omega_x+\overline{M}_y^\delta r\cdot\delta_r+\overline{M}_y^\delta a\cdot\delta_a; \\
&\dot{\omega}_x+C\omega_y\omega_z=\overline{M}_x^\beta\cdot\beta+\overline{M}_x^{\bar{\omega}}x\cdot\omega_x+\overline{M}_x^{\bar{\omega}}y\cdot\omega_y+\overline{M}_x^\delta a\cdot\delta_a+\overline{M}_x^\delta r\cdot\delta_r.
\end{aligned}
\right\}
\qquad (1.19)
$$

$$
\dot{V}=-(c_x-c_p)q\,\frac{S}{m}-g\sin\vartheta. \qquad (1.20)
$$

In the system of equations (1.19) we use the time differentiation as the point and the aerodynamic forces and moments appear as the basic linear terms of the series expansion for the parameters of motion of the aircraft. In this case we take the following definitions:

$$
\left.
\begin{aligned}
&\overline{Y}^\alpha=\frac{c_y^\alpha qs}{Vm}; \quad \overline{Y}^\delta e=\frac{c_y^{\delta e}qs}{Vm}; \quad \overline{M}_z^\alpha=\frac{m_{zb}^\alpha qsl}{J_z}; \\
&\overline{M}_z^\delta e=\frac{m_{zb}^{\delta e}qsl}{J_z}; \quad \overline{M}_z^{\bar{\omega}}z=\frac{m_z^{\bar{\omega}z}qsl^2}{2VJ_z}; \quad \overline{M}_z^{\bar{\dot{\alpha}}}=\frac{m_{zb}^{\dot{\alpha}}qsl^2}{2J_zV} \\
&\overline{Z}^\beta=\frac{c_z^\beta qs}{Vm}; \quad \overline{Z}^\delta r=\frac{c_z^\delta rqs}{Vm}; \quad c_p=\frac{P_{eng.}}{s\cdot q}, \\
&\overline{M}_y^\beta=\frac{m_y^\beta qsl}{J_y}; \quad \overline{M}_y^{\bar{\omega}}y=\frac{m_y^{\bar{\omega}y}qsl^2}{2J_yV} \quad \text{etc}; \\
&\overline{M}_x^\beta=\frac{m_x^\beta qsl}{J_x}; \quad \overline{M}_x^{\bar{\omega}}x=\frac{m_x^{\bar{\omega}x}qsl^2}{2J_xV} \quad \text{etc}; \\
&\overline{M}_y^\delta r=\frac{m_y^{\delta r}qsl}{J_y}; \quad \overline{M}_x^\delta a=\frac{m_x^{\delta a}qsl}{J_x} \quad \text{etc}; \\
&A=\frac{J_y-J_x}{J_z}; \quad B=\frac{J_z-J_x}{J_y}; \quad C=\frac{J_z-J_y}{J_x}.
\end{aligned}
\right\}
\qquad (1.21)
$$

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Expression (1.21) includes the derivatives of the aerodynamic coefficients m_{zb}^{α}, $m_{zb}^{\overline{\omega}z}$, etc., which are determined through use of the following formulas:

$$m_{zb}^{\alpha} \doteq m_z^{\alpha} \frac{b_A}{l};$$

$$m_{zb}^{\delta_e} \doteq m_z^{\delta_e} \frac{b_A}{l};$$

$$m_{zb}^{\omega z} = m_z^{\overline{\omega}z} \frac{2b_A^2}{l^2};$$

$$m_{zb}^{\dot{\alpha}} = m_z^{\dot{\alpha}} \frac{2b_A^2}{l^2}.$$

(1.22)

The necessity of such a conversion of the derivatives of the longitudinal stability is due to the fact that the quantity l is taken, rather than b_A, in the equations of spatial motion as the characteristic linear scale, as was done in computing the derivatives of the stability for an isolated longitudinal motion and in analyzing the results of wind-tunnel experiments.

4. Equations of Motion of an Aircraft in Dimensionless Form

To obtain a good generality of the results from the equation of motion of an aircraft it is advisable to convert to dimensionless form. To reduce the equations to dimensionless form let us introduce the following variables[*]:

time scale τ_m;
relative aircraft density μ;
dimensionless angular velocities $\overline{\omega}_j$;
dimensionless inertial moments of the aircraft i_j;

$$\tau_m = \frac{m}{\varrho S V}; \quad \mu = \frac{2m}{\varrho S l}; \quad \overline{\omega}_j = \frac{\omega_j l}{2V}; \quad i_j = \frac{J_j}{m\left(\frac{l}{2}\right)^2}.$$

(1.23)

Let us note that the parameters τ_m, μ and the coefficient $l/2V$, used in computing $\overline{\omega}_j$, coincide with the analogous parameters used in reducing the equations of lateral motion to dimensionless form (see, for example, [18], [19] and [34]).

---

[*]To reduce the total system of equations to dimensionless form, let us use the value of the wing span l in the future as the characteristic scale.

In transforming the equations of longitudinal motion to dimensionless form the values τ_m and μ are usually computed by using the mean aerodynamic chords (b_A) as the characteristic scale, which must be borne in mind in analyzing the total system of equations of motion cited below.

Using relationships (1.23), let us transform Equation (1.19) /23 into dimensionless form. Transformation of Equations (1.19) to dimensionless form can be carried out for the general case of aircraft flight at a constant height (ρ = const) by taking into account the variable flying speed. Change in the flying speed is described by Equation (1.20) in which the value (c_x-c_p) for simplification of analysis is assumed to be constant. Let us introduce the dimensionless time τ using the following differential equation:

$$dt = \tau_m \, d\tau, \qquad\qquad (1.24)$$

where the value τ_m is a variable due to change in the value of the flying speed V. In carrying out the transformation we must consider the following rule of differentiation:

$$\frac{d}{dt}(\omega_l) = \frac{d}{dt}\left(\bar{\omega}_l \frac{2V}{l}\right) = \frac{2V}{l}\frac{d\bar{\omega}_l}{dt} + \frac{2}{l}\frac{dV}{dt}\cdot\bar{\omega}_l. \qquad (1.25)$$

As a result of the simple transformations, and taking Equations (1.24) and (1.25) into account, we find the equations of motion for an aircraft in dimensionless form (the bar indicates differentiation by dimensionless time τ):

$$
\left.
\begin{aligned}
&\alpha' - \mu\bar{\omega}_z + \mu\beta\bar{\omega}_x = \left[-\frac{c_y^\alpha}{2} + \frac{(c_x-c_p)}{2}\right]\alpha + \\
&\quad + \frac{2g_0\tau_m^2}{l\mu}\cos\vartheta\cdot\cos\gamma - \frac{c_y^{\delta e}}{2}\delta_e + \frac{c_x-c_p}{2}\alpha_r; \\
&\bar{\omega}_z' + A\mu\bar{\omega}_x\omega_y = \bar{m}_{zb}^\alpha\cdot\alpha + \left(\bar{m}_{z0}^{\omega_z} + \frac{c_x-c_p}{2}\right)\bar{\omega}_z + \\
&\quad + \bar{m}_{zb}^{\dot{\alpha}}\cdot\frac{\alpha'}{\mu} + \bar{m}_{zb}^{\delta e}\delta_e \\
&\beta' - \mu\bar{\omega}_y - \mu\bar{\omega}_x(\alpha + \alpha_r) = \left[\frac{c_z^\beta}{2} + \frac{(c_x-c_p)}{2}\right]\beta + \\
&\quad + \frac{c_z^{\delta r}}{2}\delta_r + \frac{2g_0\tau_m^2}{l\mu}\cos\vartheta\cdot\sin\gamma;
\end{aligned}
\right\}
\qquad (1.26)
$$

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$$\bar{\omega}'_y - B\mu\bar{\omega}_x\bar{\omega}_z = \bar{m}_y^\beta\beta + \left(\bar{m}_y^{\bar{\omega}_y} + \frac{c_x - c_p}{2}\right)\bar{\omega}_y +$$

$$+ \bar{m}_y^{\bar{\omega}_x} \cdot \bar{\omega}_x + \bar{m}_y^\delta\delta_r + \bar{m}_y^{\dot{\delta}\dot{a}} \cdot \dot{\delta}_a;$$

$$\bar{\omega}'_x + C\mu\bar{\omega}_y\bar{\omega}_z = \bar{m}_x^\beta\beta + \left(\bar{m}_x^{\bar{\omega}_x} + \frac{c_x - c_p}{2}\right)\bar{\omega}_x +$$

$$+ \bar{m}_x^{\bar{\omega}_y}\bar{\omega}_y + \bar{m}_x^{\delta a} \cdot \delta_a + \bar{m}_x^{\delta r} \cdot \delta_r.$$
(1.26)

In the system of Equations (1.26) we use the following defini- tions:

$$\bar{m}_{z\,b}^\alpha = \frac{m_z^\alpha b}{i_{z\,b}} \quad \text{etc} \quad ; \quad i_{z\,b} = \frac{J_z}{m\left(\frac{l}{2}\right)^2};$$

$$\bar{m}_y^\beta = \frac{m_y^\beta}{i_y} \quad \text{etc} \quad ; \quad i_y = \frac{J_y}{m\left(\frac{l}{2}\right)^2};$$
(1.27)

$$\bar{m}_x^\beta = \frac{m_x^\beta}{i_x} \quad \text{etc} \quad \cdot \quad i_x = \frac{J_x}{m\left(\frac{l}{2}\right)^2}.$$

For many problems in the dynamics of an aircraft, the effect of gravitational forces on the disturbed motion of an aircraft relative to the center of mass is small. In particular, this refers to the case when short-range motion is analyzed. However, the gravitational forces determine the longitudinal trim of an aircraft and therefore must be taken into account in the original undisturbed motion. In this latter case, the form of Equations (1.26) is practically constant, only $\alpha_{h.f.}$ (the angle of attack in horizontal flight) must be added to α_m, and we must take into account that α and δ_e are increases in the respective variables with respect to their values in horizontal flight.

In view of the fact that in the future we shall analyze the equations of motion in dimensionless form at a constant flying speed, when $c_x = c_p$, let us rewrite them again taking the comments made above into account and omitting the gravitational terms:

$$\alpha' - \mu\bar{\omega}_z + \mu\beta\bar{\omega}_x = -\frac{c_y^\alpha}{2}\alpha - \frac{c_y^\delta e}{2}\delta_e;$$

$$\bar{\omega}_z' + A\mu\bar{\omega}_x\bar{\omega}_y = \bar{m}_{z_\mathrm{b}}^\alpha a + \bar{m}_{z_\mathrm{b}}^{\bar{\omega}_z}\bar{\omega}_z +$$

$$+ \frac{\bar{m}_z^{\dot{\alpha}}\mathrm{b}}{\mu}\alpha' + \bar{m}_z^\delta{}_\mathrm{b}^e \cdot \delta_e;$$

$$\beta' - \mu\bar{\omega}_y - \mu\bar{\omega}_x(\alpha_\mathrm{m} + \alpha) = \frac{c_z^\beta}{2}\beta + \frac{c_z^\delta r}{2}\delta_r;$$

$$\bar{\omega}_y' - B\mu\bar{\omega}_x\bar{\omega}_z = \bar{m}_{y'}^\beta \cdot \beta + \bar{m}_y^{\bar{\omega}_y}\bar{\omega}_y +$$

$$+ \bar{m}_y^{\bar{\omega}_x} \cdot \bar{\omega}_x + \bar{m}_y^\delta r \cdot \delta_r + \bar{m}_y^\delta a \cdot \delta_a;$$

$$\bar{\omega}_x' + C\mu\bar{\omega}_y\bar{\omega}_z = \bar{m}_x^\beta \cdot \beta + \bar{m}_x^{\bar{\omega}_x}\bar{\omega}_x +$$

$$+ \bar{m}_x^{\bar{\omega}_y} \cdot \bar{\omega}_y + \bar{m}_x^{\bar{\omega}_x} \cdot \bar{\omega}_x + \bar{m}_x^\delta a \cdot \delta_a + \bar{m}_x^\delta r \cdot \delta_r;$$

(1.28)

$$\dot{V} \cong -(c_x - c_p)\frac{qs}{m} = 0.$$

(1.29)

The convenience of the dimensionless form of the equations con-
sists especially in the fact that without solving Equations (1.26)
we can make several general conclusions. In particular, from the
equations it follows that the value μ is the only parameter in the
system of Equations (1.26) (if $\alpha_\mathrm{h.f.}$ = const) which depends on the
flight path. The value μ is determined by height and is independent
of flying speed. Hence it follows that the qualitative picture of
the aircraft motion acted on by disturbances, and with control,
depends only on height of flight and is independent of speed. The
effect of the height of flight on the qualitative picture of motion
is manifested only through the dependence of the aerodynamic coef-
ficients on the M number (or through change in $\alpha_\mathrm{h.f.}$). The basic
effect of the flying speed is expressed in the change in time scale.
For example, when $\alpha_\mathrm{h.f.}$ = const, the frequency of oscillations and
damping time depend on flying speed, but the stability of motion,
the value of the dimensionless decrement of damping, the number and
forms of the points of rest of the system of equations of motion,
etc., are independent thereof (this is all true with the single
stipulations that the aerodynamic coefficients are independent of
the flying speed).

Equations (1.26) are nonlinear equations with constant coefficients even for the case of aircraft motion at a variable speed. In this case the variability of speed is manifested in the change in the coefficients of damping due to the existence of the coefficient $(c_x - c_p)$ and in the change in time scale. With transition to dimensional time we must use the following dependence of t on τ:

$$t = \int_0^\tau \tau_m \, d\tau = \frac{m}{\varrho s} \int_0^\tau \frac{d\tau}{V(\tau)}. \qquad (1.30)$$

To find the dependence between t and τ in explicit form, let us transform Equation (1.29) into a new variable τ:

$$\frac{dV}{d\tau} \cong -\frac{(c_x - c_p)}{2} V. \qquad (1.31)$$

After integrating Equation (1.31) with $(c_x - c_p)$ = const, we find:

$$V = V_0 e^{-\left(\frac{c_x - c_p}{2}\right)\tau}. \qquad (1.32)$$

And finally by substituting (1.32) into (1.30) and integrating we find the formula for the dependence of t on τ:

$$t = \frac{m}{\varrho s V_0} \frac{2}{(c_x - c_p)} \left[e^{\left(\frac{c_x - c_p}{2}\right)\tau} - 1 \right]. \qquad (1.33)$$

It is easy to prove that when $(c_x - c_p) \to 0$, Formula (1.33) in the limit converts to the identity $(t \equiv \tau \cdot \tau_{m0})$.

From the system of Equations (1.26) it is immediately clear that the acceleration of the aircraft (flight when $c_p > c_x$) increases the damping of the oscillations, i.e., it affects the stabilizing oscillations, and the drag $(c_p < c_x)$ (destabilizing oscillations), i.e., with flight at a decreasing speed the effective damping also decreases. All these conclusions are found for the equations of motion that are practically precise, in particular the equations of short-range longitudinal motion and the equations of lateral motion. In these cases the solutions for the motion of an aircraft during flight at variable speed can be written in explicit form.

5. Approximate Classification of the Motions of an Aircraft, About the Longitudinal, Lateral and Vertical Axes .

Before we begin a study of the features of spatial motion of an aircraft, let us bear in mind several results which pertain to the so-called isolated longitudinal and lateral motions.

Let us look, as an example of the initial undisturbed motion, at the steady horizontal flight of an aircraft without rolling (V_0 = const, H_0 = const, $\Theta_0 \simeq 0$) and study the disturbed motion only over a short segment of time. In this case we can ignore the change in flying speed ($dV/dt \simeq 0$). As was shown above, with small disturbances, the motion of an aircraft in its pitching plane (about the axis OZ_1) does not depend on its motion relative to the axes OX_1 and OY_1, and the motion about the axes OX_1 and OY_1 does not depend on the motion about the axis OZ_1. This means that to study the properties of motion of an aircraft during small disturbances we can look separately at the isolated logitudinal and isolated lateral motions, which are described respectively by the following two groups /27 of equations:

$$\alpha' = \mu \bar{\omega}_z - \frac{c_y^\alpha}{2}\alpha - \frac{c_y^{\delta_e}}{2}\delta_e;$$

$$\bar{\omega}_z' = \overline{m}_{z_\mathrm{b}}^\alpha \alpha + \overline{m}_{z_\mathrm{b}}^{\omega_z}\bar{\omega}_z + \frac{\overline{m}_{z_\mathrm{b}}^{\dot\alpha}}{\mu}\alpha' + \overline{m}_{z_\mathrm{b}}^{\delta_e}\cdot\delta_e;$$

$$\vartheta' = \mu\bar{\omega}_z;$$

$$\hspace{3cm} (1.34)$$

$$\beta' = \mu\bar{\omega}_y + \mu\bar{\omega}_x(\alpha_m + \alpha_{\mathrm{h.f.}}) + \frac{c_z^\beta}{2}\beta +$$

$$+ \frac{c_z^{\delta_r}}{2}\delta_r + \frac{2g_0\tau_m^2}{l\mu}\sin\gamma;$$

$$\bar{\omega}_y' = \overline{m}_y^\beta\cdot\beta + \overline{m}_y^{\omega_y}\cdot\bar{\omega}_y + \overline{m}_y^{\omega_x}\cdot\bar{\omega}_x + \overline{m}_y^{\delta_r}\cdot\delta_r + \overline{m}_y^{\delta_a}\cdot\delta_a;$$

$$\bar{\omega}_x' = \overline{m}_x^\beta\cdot\beta + \overline{m}_x^{\omega_x}\cdot\bar{\omega}_x + \overline{m}_x^{\delta_a}\cdot\delta_a + \overline{m}_x^{\delta_r}\cdot\delta_r;$$

$$\gamma' = \mu\bar{\omega}_x.$$

$$\hspace{3cm} (1.35)$$

The properties of the solutions to these two systems of equations have been analyzed in detail in many papers (see, for example, [18], [19], [34]), therefore we shall look at them only insofar as they are needed for further discussion.

A. Isolated Longitudinal Motion.

Equations (1.34) can be easily reduced to a single equation of second order, which describes the change in the increase in the angle of attack of an aircraft (the equation for ω_z has an analogous form):

$$\alpha'' + \left(\frac{c_y^\alpha}{2} - \overline{m}_z^{\omega_z} \overline{b} - \overline{m}_z^{\overline{\alpha}} \overline{b} \right) \alpha' + \mu \left(-\overline{m}_z^\alpha \overline{b} - \frac{c_y^\alpha \, \overline{m}_z^{\omega_z}}{2\mu} \overline{b} \right) \alpha =$$
$$= \mu \left(\overline{m}_z^{\delta_e} \overline{b} + c_y^{\delta_e} \frac{\overline{m}_z^{\omega_z}}{2\mu} \overline{b} \right) \delta_e - \frac{c_y^{\delta_e}}{2} \cdot \delta_e' . \tag{1.36}$$

The following inequalities are the conditions for the aperiodic and oscillational stability of the solutions to Equation (1.36):

$$\left(-\overline{m}_z^\alpha \overline{b} - c_y^\alpha \frac{\overline{m}_z^{\omega_z}}{2\mu} \overline{b} \right) > 0; \tag{1.37}$$

$$\left(\frac{c_y^\alpha}{2} - \overline{m}_z^{\omega_z} \overline{b} - \overline{m}_z^{\overline{\alpha}} \overline{b} \right) > 0. \tag{1.38}$$

The solution to Equation (1.36) for an aircraft possessing /28 static stability, usually has a form corresponding to the oscillational character of the motion and can be written in the form:

$$\alpha(\tau) = e^{-\xi\tau} \cdot A_1(\alpha(0), \alpha'(0)) \cdot \cos[\overline{\omega}_0 \tau + \varphi_0(\alpha(0), \alpha'(0)], \tag{1.39}$$

where

$$\xi = \frac{1}{2} \left(\frac{c_y^\alpha}{2} - \overline{m}_z^{\omega_z} \overline{b} - \overline{m}_z^{\overline{\alpha}} \overline{b} \right);$$

$$\overline{\omega}_0 = \sqrt{\mu \left(-\overline{m}_z^\alpha \overline{b} - \frac{c_y^\alpha}{2\mu} \overline{m}_z^{\omega_z} \overline{b} \right) - \xi^2};$$

A_1 and ϕ_0 are arbitrary constants, depending on the initial conditions $\alpha(0)$ and $\alpha'(0)$.

Since the value ξ is usually not high, the disturbed motion by angle of attack then has an oscillational character with an

oscillation period determined by the following formula:

$$\tau_\alpha = \frac{2\pi}{\omega_0}.$$

After conversion to dimensional values the approximate expression for the period of longitudinal oscillations T_α can be represented in the form:

$$T_\alpha \simeq 2\pi \sqrt{\frac{J_z}{-m_z^\alpha q s b_A}}. \qquad (1.40)$$

As follows from Equation (1.36), with deflection of the elevator, after damping of the transient condition, the angle of attack of the aircraft varies by the value $\Delta\alpha_b$, which is approximately equal to

$$\Delta\alpha_b \simeq \frac{-m_z^{\delta e} \cdot \bar{\lambda}_e}{m_z^\alpha}. \qquad (1.41)$$

B. Isolated Lateral Motion.

From the system of Equations (1.35) it follows that the isolated lateral motion of an aircraft with small disturbances is described by a system of differential equations of fourth order, and consequently depends on the four roots of the characteristic equation. The characteristic equation usually contains a pair of complex-conjugate roots, which represent the yawing motion of the aircraft; one negative root, which is large in absolute value, represents the rolling motion and one real root, which in the general case is positive and small in absolute value, is negative and represents the so-called spiral motion of the aircraft, caused in first order by the development of side slip due to the effect of gravity during rolling. Since we are interested in the future in rather rapid motions, the spiral motion of an aircraft, which is slow, will not be studied and in Equations (1.35) we have set $g_0 = 0$. In this simplification the slow rolling motion of the aircraft is not taken into account and the aircraft is neutrally stable with respect to rolling.

To find the simple qualitative relationships let us make additional simplifications of Equations (1.35), i.e., let us set the following coefficients equal to zero:

$$\left. \begin{array}{l} (\alpha_r + \alpha_{h.f.}) = \overline{m}_y^{\bar{\omega}x} = 0; \\ \overline{m}_x^{\delta r} = \overline{m}_y^{\delta a} = 0. \end{array} \right\} \qquad (1.42)$$

With such simplifications the solution to the equation for change in the angle of side slip of an aircraft is independent of the angular velocity and the angle of bank, and the equation itself can be written in the form:

$$\beta'' + \left(-\frac{c_z^\beta}{2} - \overline{m}_y^{\overline{\omega}_y} \right) \beta' + \mu \left(-\overline{m}_y^\beta + \frac{c_z^\beta \cdot \overline{m}_y^{\overline{\omega}_y}}{2\mu} \right) \beta =$$
$$= \mu \left(\overline{m}_y^\delta r - \frac{c_z^\delta r \overline{m}^{\overline{\omega}_y}}{2\mu} \right) \delta_r + \frac{c_z^\delta r}{2} \cdot \delta'_r \qquad (1.43)$$

i.e., it is fully analogous to the equation for the isolated longitudinal motion (1.36).

The conditions of the aperiodic and oscillational stability of the solutions are written exactly as (1.37) and (1.38):

$$\left(-\overline{m}_y^\beta + \frac{c_z^\beta \cdot \overline{m}_y^{\overline{\omega}_y}}{2\mu} \right) > 0; \qquad (1.44)$$

$$\left(-\frac{c_z^\beta}{2} - \overline{m}_y^{\overline{\omega}_y} \right) > 0. \qquad (1.45)$$

The yawing motion of an aircraft has an oscillational character with an oscillation period determined by the following approximate formula, written in dimensional form:

$$T_\beta \simeq 2\pi \sqrt{\frac{J_y}{-m_y^\beta q s l}}. \qquad (1.46)$$

We can approximately look also at the isolated disturbed rolling motion of an aircraft which is written by satisfying Equations (1.42) in the following simplified equation:

$$\overline{\omega}'_x - \overline{m}_x^{\overline{\omega}_x} \cdot \overline{\omega}_x = \overline{m}_x^\beta \cdot \beta + \overline{m}_x^\delta a \delta_a \qquad (1.47)$$

The solution to Equation (1.47) when $\beta = \delta_a = 0$ is represented in the form of an exponential time function and is stable in those cases when $\overline{m}_x^{\overline{\omega}_x} < 0$.

The results which pertain to the properties of isolated longitudinal and lateral motions, are fully sufficient for further

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studies of spatial motion.

6. Characteristics and Basic Problems of the Dynamics of Motion of Modern Aircraft.

With what then are the characteristics of spatial motion of an aircraft associated in comparison with the isolated motions? Mathematically these characteristics are due to the equations in the general case of spatial motion not being separated since the motions of an aircraft relative to the major inertial axes OX_1, OY_1 and OZ_1 are interrelated. If the motion relative to the axis OZ_1 is determined to be longitudinal and relative to the axes OX_1 and OY_1 to be lateral then with spatial motion we can speak of an interaction or interrelationship between the longitudinal and lateral motions. The reasons for such an interaction are several and all of them exert some influence on the motion of the aircraft. We can determine the following basic forms of the interaction of longitudinal and lateral motions of an aircraft: aerodynamic, kinematic, inertial and also the interaction caused by the effect of the gyroscopic moment of the engine.

A. Aerodynamic Interaction

By the aerodynamic interaction we mean the effect of the dependence of the aerodynamic derivatives of the stability of the lateral motion on the parameters of the longitudinal motion, in first order on the angle of attack α, and the aerodynamic derivatives of the longitudinal motion on the parameters of the lateral motion (for example, on the angle of side slip β).

Above (see Section 3) the basic system of equations of disturbed motion was simplified by linearization. The linearization was carried out relative to the parameters of the original steady and straight flight without side slip. The retention in the expansion of only first order terms of smallness was one of the conditions for the existence of separation of the equations. However, as noted above, the dependence of the aerodynamic forces and moments on the parameters of motion in fact has a more complex character, especially with nonsymmetric flight (for example, during flight involving side slip) or the presence of rather large disturbances. In this latter case we must take into account the dependence of the rolling moment and the yawing moment not only on the angle of side slip, angular rolling and yawing velocities, but also on the angle of attack. Thus, for example, in studying the rolling moment (or its coefficient) we took in Section 3

$$m_x = m_x^{\beta}\beta + m_x^{\delta_a}\delta_a + m_x^{\bar{\omega}_y} \cdot \bar{\omega}_y + m_x^{\bar{\omega}_x}\bar{\omega}_x.$$

In studying the original system $\beta_0 = 0$, under the condition of small disturbances, such an approximation for the rolling moment

corresponds to the real picture to a sufficient degree. In the same case, when the disturbances reach large values, we can never ignore the effect of the angle of attack on the rolling moment in the presence of the angle of side slip, since in certain instances variations in the angle of attack may change even the sign of the rolling moment during side slip. Figure 1.4 shows examples of the dependences of the derivatives of the moments of roll and yaw, according to the angle of side slip, on the angle of attack. From Figure 1.4 it is obvious that a more precise approximation of the rolling moment may be obtained if we introduce the following relationships:
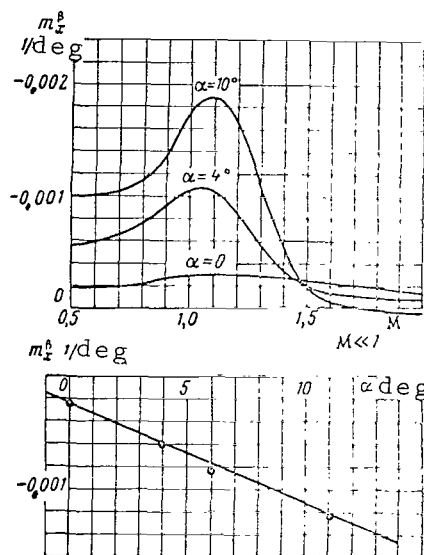
Fig. 1.4.

$$m_x^\beta = m_{x0}^\beta + \frac{\partial m_x^\beta}{\partial \alpha}\alpha.$$

Analogous dependences can be cited for the coefficients of the yawing moment (Fig. 1.5) and the coefficient of aileron efficiency; the coefficients of the lateral force depend less on the angle of attack and in such case such precision is virtually not required.

For the coefficients of the longitudinal moment, taking into account the disturbances which are insignificant in value may also produce the necessity for introducing approximations which allow for the effect of side slip. Such an approximation may be made only approximately,

$$m_z = m_z^\alpha \alpha + c_1 \beta^2 \alpha;$$

and analogously for the lift

$$c_y = c_y^\alpha \alpha + c_2 \beta^2 \alpha.$$

However in practice the effect of side slip on the derivatives m_z^α and c_y^α is small.

In allowing for the dependences mentioned above, the principles /32 of separating the system of equations of the disturbed motion into two independent systems, which describe the longitudinal and the lateral disturbed motions, is disrupted. Thus, the more accurate approximation of the aerodynamic coefficients indicates the presence of an aerodynamic interaction between the lateral and the longitudinal

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motions. The terms of the equations which determine its interaction
are nonlinear. Consequently, in the general case account is taken
of the aerodynamic interaction, thus leading to the necessity of
solving the system of nonlinear equations that is necessary to do
even if computers are used. In the following sections we shall
look at several problems where the manifestation of the aerodynamic
interaction is substantial. In each, if it is taken into account,
we shall make special stipulations.
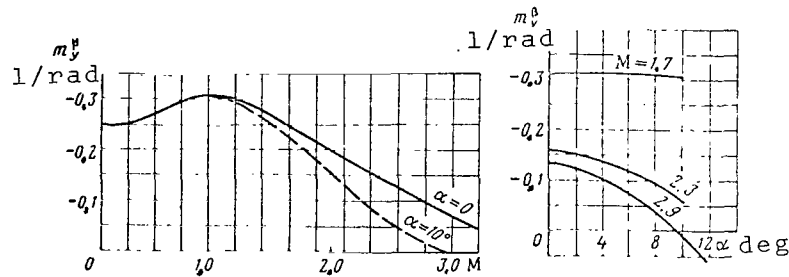
Fig. 1.5.

B. Kinematic Interaction

When an aircraft enters strongly into a roll, when its axis
OX_1 practically retains an unchanged position in space, a simul-
taneous change in the angle of attack and the angle of side slip
of the aircraft occurs. This is due to the appearance of a kine-
matic interaction of motions. To clarify this type of interaction,
let us look at a simplified diagram of the motion of an aircraft
during rolling.

We shall assume that the axis of the aircraft OX_1 and the
vector of the flying speed $\overline{V}$ retain a constant position in space
and the aircraft begins to change the angle of bank. Then during
rolling $\gamma = 90°$. the angle of attack of the aircraft α_0 "changes"
to the angle of side slip β, and when $\gamma = 180°$, $\beta = 0$, $\alpha = -\alpha_0$,
etc. (Fig. 1.6). The terms which determine the kinematic inter-
action enter into the equations for α' and β'.

Let us obtain the results written above from the equations of
motion. For this, in the equations for α and β, let us omit all
terms which determine the forward motion of the OX_1 axis, i.e.,
let us set equal to zero all terms other than $\beta\overline{\omega}_x\mu$ and $-\alpha\overline{\omega}_x\mu$. We   /33
find the following system of equations:

$$
\left.
\begin{aligned}
\alpha' + \beta\left(\overline{\omega}_x\mu\right) &= 0; \\
\beta' - \alpha\left(\overline{\omega}_x\mu\right) &= 0; \\
\gamma' &= \mu\overline{\omega}_x.
\end{aligned}
\right\}
\qquad (1.48)
$$

Instead of $\mu\bar{\omega}_x$, let us substitute the value γ' equal to it, and transform Equations (1.48) into the form

$$\left.\begin{array}{l} \alpha' + \beta\gamma'(\tau) = 0; \\ \beta' - \alpha\gamma'(\tau) = 0. \end{array}\right\} \qquad (1.49)$$

It is easy to find direct proof that the solution to the system of Equations (1.49) for the arbitrary function $\gamma(\tau)$ and the initial conditions $\alpha(0) = \alpha_0$, $\beta(0) = 0$ has the following form:

$$\left.\begin{array}{l} \alpha(\tau) = \alpha_0 \cos\gamma; \\ \beta(\tau) = \alpha_0 \sin\gamma, \end{array}\right\} \qquad (1.50)$$

i.e., α and β in the assumptions made above concerning the constancy of the position of the OX_1 axis in space are periodic functions of the angle of bank γ.
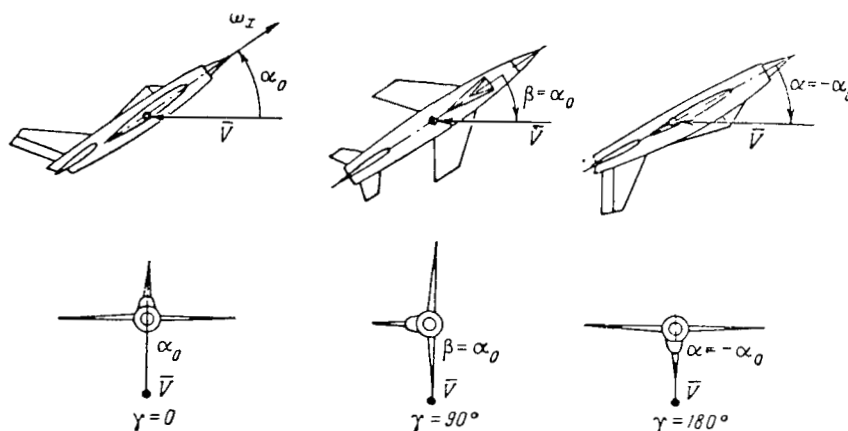
Fig. 1.6.

In the general case the OX_1 axis under the condition of rolling of the aircraft is shifted and rotated in space and the changes in the angles of attack and side slip have a more complex character. However, at the beginning of the motion, and also during turning of the aircraft at very high angular velocities of rolling ω_x, relationships (1.50) qualitatively describe the changes in the angles of attack and side slip quite well.

C. Inertial Interaction

In the equations of the moments (the equations for the deriva- /34 tives $\bar{\omega}_z'$, $\bar{\omega}_y'$, $\bar{\omega}_x'$) enter the terms which contain the products of the angular velocities of the type $A\bar{\omega}_x \cdot \bar{\omega}_y$, $B\bar{\omega}_x \cdot \bar{\omega}_z$ and $C\bar{\omega}_y \cdot \bar{\omega}_z$. These moments can be conditionally termed moments of inertial interaction

or gyroscopic moments. The physical meaning of these terms in the equations of motion consists in that they take into account the appearance of centrifugal forces of inertia during the turning of an aircraft relative to the axis which does correspond to the major industrial inertial axis. Let us look, for example, at the appearance of the moment $(J_y - J_x)\omega\dot{y}\cdot\omega_x$, which acts on the aircraft relative to the OZ_1 axis. When the aircraft turns at angular velocities ω_y and ω_x, the moment from the inertial forces acts on it and can be approximately determined in the following manner. Let us look at a simplified model of an aircraft, i.e., let us assume that its entire mass is concentrated in two equal loads M_1 and M_2, distributed at a distance l_x from the center of mass. The motion of the aircraft at angular velocities ω_y and ω_x are equivalent to its rotation relative to the vector $\overline{\Omega}$ (Fig. 1.7). In this case the centrifugal forces, which are found using the relationships that are known from mechanics,
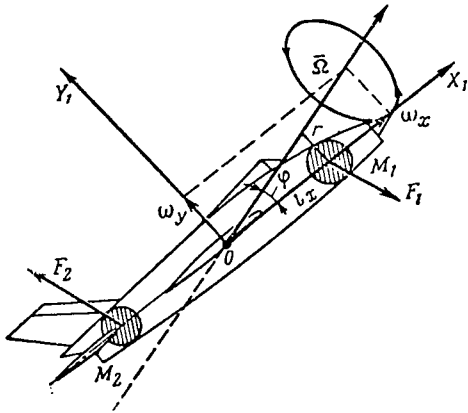
Fig. 1.7.

$$F_1 = M_1 r \Omega^2 = M_1 l_x \sin\varphi \cdot \Omega^2;$$
$$F_2 = M_2 r \Omega^2 = M_2 l_x \sin\varphi \cdot \Omega^2.$$

act on the forces M_1 and M_2. The expression for the moment from these forces is found after multiplying the values of the forces by the respective arms:

$$M_z^{in} = F_1 l_x \cos\varphi + F_2 l_x \cos\varphi.$$

Taking into account that $2M_1 l_x^2 = J_y - J_x$ and

$$\Omega \sin\varphi = \omega_y; \quad \Omega \cos\varphi = \omega_x,$$

we find the final expression for the inertial moment relative to the OZ_1 axis:

$$M_z^{in} = (J_y - J_x)\omega_y\omega_x.$$

$$(1.51)$$

Analogously we can find the expressions for the inertial moments relative to the axes OX_1 and OY_1 (see, Fig. 1.8):

$$M_y^{in} = (J_z - J_x)\omega_z\omega_x;$$

$$(1.52)$$

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$$M_x^{in} = (J_z - J_y)\,\omega_y\omega_z. \qquad (1.53)$$
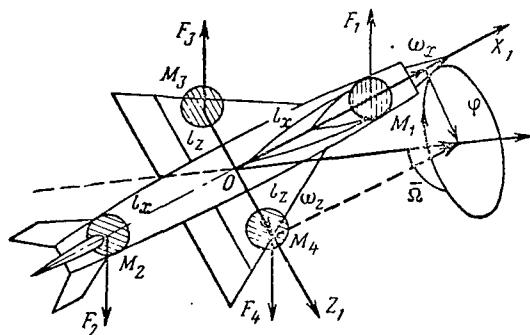
Fig. 1.8.

From expressions (1.51) - (1.53) it follows that the appearance of the inertial moment relative to each of the major inertial axes is due to the angular velocities of the aircraft motion relative to the other two orthogonal axes. Thus, the motion of an aircraft relative to all these major inertial axes in the general case become interrelated if the vector of the angular velocity does not correspond to any of these axes. In this case, since during the maneuvering of an aircraft the greatest angular velocity is usually the angular rolling velocity, then as follows from the formulas given above, the greatest influence is exerted by the inertial moments M_z^{in} and M_y^{in}.

The effect of the inertial interaction on the dynamics of an aircraft in carrying out rolling maneuvers is quite substantial. To a large degree the characteristics of the spatial motion of an aircraft are due mainly to this type of interaction.

D. The Effect of the Gyroscopic Moment of the Engine

For completeness of the picture let us say several words concerning the interaction, caused by the presence of the rotating rotor of the engine. The rotating rotor of the engine with the kinetic moment $(\overline{J_{eng} \cdot \omega_{eng}})$ represents a gyroscope, and as any gyroscope in response to a certain external moment applied on it causing it to turn at an angular velocity $\overline{\omega}$, it will tend to precess in an orthogonal direction or will create a moment equal to

$$\overline{M}_{gyr} = [\overline{J_{eng}\omega_{eng} \cdot \overline{\omega}}] \qquad (1.54)$$

To take into account the effect of the gyroscopic moment of the engine, in the right-hand sides of the equations of spatial motion of the aircraft it follows to add to the equation for $\dot{\omega}_y$ the term

$$\frac{J_{eng}\omega_{eng} \cdot \omega_z}{J_y}$$

and to the equation for $\dot{\omega}_z$, the term

$$\frac{J_{eng}\omega_{eng} \cdot \omega_y}{J_z}$$

The presence of the gyroscopic moment of the engine leads to the appearance also of yaw when the pitching maneuver is carried out by the aircraft, and when the angle of yaw is varied the angle of attack begins simultaneously to be varied (Fig. 1.9). In the figure the arrows show the direction of motion of the aircraft nose during
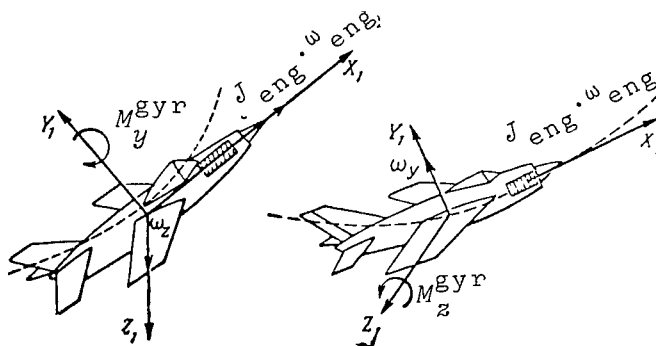
Fig. 1.9.

the maneuver under the influence of the gyroscopic moment of the engine. During rolling maneuvers the existence of the gyroscopic moment of the engine leads to a slight nonsymmetry in the motion of the aircraft as a function of the direction of rolling (port or starboard). However, the effect of the gyroscopic moment of the engine on the motion of the aircraft within the framework of the questions studied in this paper leads to a quantitative change and further on in this paper we will not take it into account. In each specific instance allowing for it presents no difficulty.

Thus, from the basic principles studied above of the interaction of the simplest forms of aircraft motion (longitudinal and lateral) the major ones are the inertial, aerodynamic, kinematic and gyroscopic interactions. In this case in the inertial interaction the shape of the ellipsoid of inertia of the aircraft is of fundamental significance. With the elongated ellipsoid of inertia, character- /37 istic of modern aircraft and other winged craft, the differences in the inertial moments $(J_y - J_x)$ and $(J_z - J_x)$ are very large values. In this latter case it is especially important to study all types of aircraft motions, accompanied by any substantial rolling. The motion of an aircraft at an angular rolling velocity, as was shown above, leads to the development of large moments of inertial interaction acting on the aircraft. In studying the motion of an aircraft this causes us to turn once more to the total equation of motion in nonlinear form, since in this case it is not allowable that the terms which contain the products of the angular velocities be omitted.

In this connection, one of the most important problems in dynamics is the study of the class of spatial motions of an aircraft with angular rolling velocity. Here we can mention any number of spatial maneuvers of an aircraft that are widely used in flight

practice:

(1)   Entry and exit from turn and chandelle;

a) "Roll" ($\gamma \sim 360°$ or greater).

b) Turn with variable direction. ($\gamma \leq 180°$)

c) Half-loop with turn. ($\gamma \sim 180°$)

d) Entry into dive with turn. ($\gamma \sim 180°$)
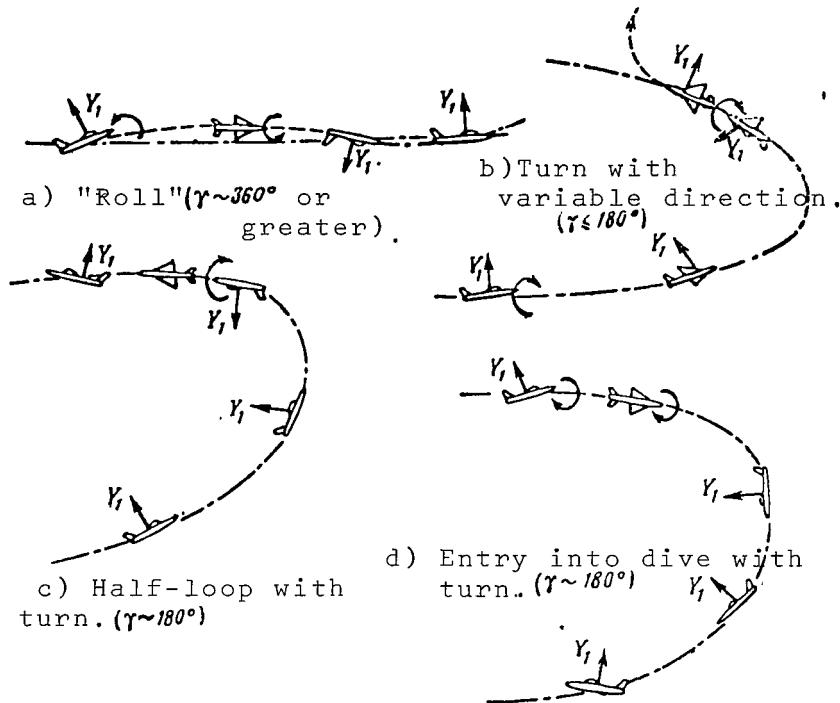
Fig. 1.10.

(2)   Split S in horizontal flight;

(3)   Immelman turns (turns with different values of the normal /38 G-force.)

(4)   Split S during entry into a dive (turn with negative normal G-force);

(5)   Snap and slow turns;

(6)   Turn with variable direction of loop (figure eight), etc. (Fig. 1.10).

From the above it follows that in the general case we must analyze the controlled motion of an aircraft with the simultaneous action of the pilot using lateral and longitudinal controls and in a number of instances directional control as well. A slight simplification in solving these complex problems may be obtained if we take into account that all the maneuvers mentioned above, as a rule, occur with a quite small change in the linear flying speed,

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which can be ignored. This assumption will be used in the future.
The next sections of the book are devoted to a description of the
different methods used and the analytical results of the classes
of aircraft motion mentioned above and several conclusions concerning
its formulation which follow from this analysis.

CHAPTER II

STABILITY OF AIRCRAFT MOTION DURING STEADY ROTATION ABOUT THE LONGITUDINAL AXIS

7. Deriving the Conditions of Stability of Aircraft Motion During Steady Rotation About the Longitudinal Axis

Before we begin an investigation of the dynamics of an aircraft in carrying out complex spatial maneuvering accompanied by rolling, let us look at the stability of its motion during rotation at a constant angular velocity relative to the longitudinal axis. Study of this special type of motion permits finding several characteristics of the aircraft motion associated with the presence of nonlinear terms in the equation of motion.
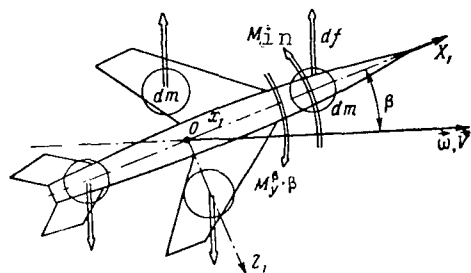
Fig. 2.1.

Let us look at a simplified physical diagram of the aircraft motion. During rotation of an aircraft about the axis that does not coincide with the major inertial axis and comprising a certain angle α with the velocity vector $\bar{V}$, in addition to the aerodynamic moment of stability acting on it there will be also the inertial moment from the centrifugal forces which can be easily computed approximately if we assume that the entire mass of the aircraft is distributed along the axes OX_1 and OZ_1. Let us assume that the static stability is high, then the rotation of the aircraft will occur relative to the velocity vector $\bar{V}$ (Fig. 2.1). Let us look at the force df, which acts on the elementary mass dm of the aircraft. It is equal to the product

$$df = \omega^2 x_1 \sin\beta\, dm.$$

If we multiply the force df by the arm $(x_1 \cos\beta)$, we find the ele- mentary moment of the inertial forces

$$dM_{in} = df x_1 \cos\beta = x_1^2 \sin\beta \cos\beta \omega^2\, dm.$$

Carrying out summation (integration) over the entire mass of the aircraft, distributed along the axes OX_1 and OZ_1, we find an expression for the total inertial moment

$$M_{y_{in}}^{\cdot} = \int_m x_1^2 \sin\beta \cos\beta\omega^2 dm - \int_m z_1^2 \sin\beta \cos\beta\omega^2 dm =$$

$$\cong (J_z - J_x)\omega^2\beta. \qquad (2.1a)$$

An analogous expression can be found if we look at the deflection of the aircraft from the angle of attack α and determine the inertial moment acting relative to the axis OZ:

$$M_z^{in} \cong (J_y - J_x)\omega^2\alpha. \qquad (2.1b)$$

Both moments, computed in this manner, are proportional to the angles of deflection (β or α) and tend to increase them. Let us look in somewhat greater detail at the yawing motion of an aircraft and assume that the degree of its pitching stability is so high that the angle of attack during the motion can be assumed to be constant.

During the rotation of an aircraft at a constant angular rolling velocity ω = const, in addition to the aerodynamic stabilizing moment, an additional destabilizing moment will act on it, the expression for which was found above, proportional to the square of the angular rolling velocity ω, which decreases the "effective" degree of static stability of the aircraft and with a high value of the angular rolling velocity will lead to a loss in motion stability. It is obvious that the losses in stability will take place when the inertial moment is greater than the stabilizing aerodynamic moment. Consequently, there exists a certain critical value of the angular rolling velocity ω, which can be determined from the condition that the aircraft during yawing motion (in analyzing the longitudinal motion, during pitching) is neutrally stable. In approximate form this condition of neutral stability is found by equating the inertial and aerodynamic moments and can be written as:

$$(J_z - J_x)\omega^2 + m_y^\beta qsl \quad 0 \qquad (2.2)$$

From this relationship there follows the approximate formula for the *critical angular rolling velocity*, which losses in stability of pitching motion of the aircraft are possible

$$\omega_\beta = \sqrt{\frac{-m_y^\beta qsl}{J_z - J}} \qquad (2.3)$$

Analogously, if the degree of pitching stability of the aircraft is much less than its yawing stability and if we assume that

during the time of the motion $\beta \simeq 0$, then the condition of neutral stability of pitching motion of the aircraft, turning in a rolling motion, can be written in the form of the equation

$$(J_y - J_x)\omega^2 + m_z^\alpha qsb_A = 0,$$

whence it is easy to find an approximate expression for the *critical angular rolling velocity*, in which losses in the stability of the pitching motion of the aircraft are possible

$$\omega_\alpha = \sqrt{\frac{-m_z^\alpha qsb_A}{J_y - J_x}} \, . \tag{2.4}$$

These expressions have been obtained on the assumption that the motion of an aircraft is a "plane one", i.e., there is included in the change either only the angle of yaw ($\alpha \sim$ const), or the angle of pitch ($\beta \sim$ const), since in these cases we can not take into account the pitching motion of an aircraft, for example, in analyzing a yawing motion. In all the discussions given above it was essential that the original motion of the aircraft be taken as the basis, in which the aircraft turns relative to the vector of angular velocity and "slips" along a certain cone, caused by the rotation of the
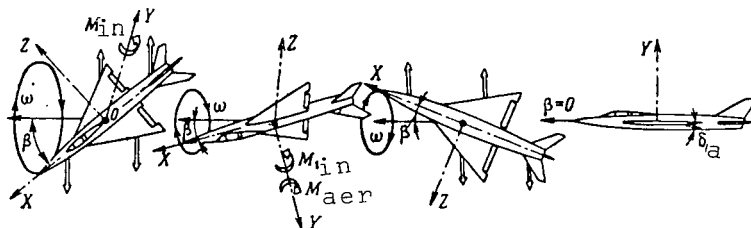
Fig. 2.2.

axis OX_1 around the velocity vector $\vec{V}$, corresponding in the example with yawing motion to a constant angle of attack α_0 (Fig. 2.2), and in the example with pitching motion, to a constant angle of side slip $\beta \sim 0$. In order for the original motion to occur, it is necessary that the value of the natural frequency of the aircraft oscillations (in the example with yawing motion this is the frequency of the longitudinal oscillations) be considerably greater than the angular rolling velocity. In this case the aircraft succeeds in reacting to the disturbances caused by the kinematic interrelationship during rolling between the angles α and β, having a frequency equal to the value of the angular rolling velocity (see Fig. 1.6).

Let us see in greater detail how the rolling motion of an air- craft develops. Let the aircraft, possessing a large reserve of longitudinal stability and trimmed at the angle of attack $\alpha_0 > 0$,

begin to roll. In such case at the first moment of time it rotates relative to the major inertial axis OX_1, which because of the kinematic interrelationship of the angles α and β will lead to the simultaneous appearance of the angle of side slip β and a slight decrease in the angle of attack α_0. Due to the large longitudinal stability of the aircraft the angle of attack α_0 begins to be recovered thus leading to the appearance of an additional angular velocity $\Delta\omega_z$ and to a deviation of the vector of the total angular velocity from the axis OX_1. As a result of such a condition the motion is regulated, in which the aircraft "rolls over" along a certain cone, the axis of which coincides with the vector $\bar{V}$, retaining the angle of attack α_0 between the plane of the wings and the velocity vector $\bar{V}$.

With a rapid rotation of the aircraft, when the angular rolling velocity significantly exceeds the natural frequencies of the oscillations in pitching and yawing, the aircraft will not be able to react to the periodic variations in the angles α and β at the frequency ω_x, caused by the kinematic interrelationship, and its rotation will take place relative to the major inertial axis OX_1. The angles of attack and side slip in such case on the average have zero values. In this case the spatial characteristics of the aircraft motion are substantial. The aerodynamic coefficients of the stability during rapid rolling turn of the aircraft no longer influence its motion to any appreciable degree. The rolling and turning motion of an aircraft at the limit, when $\omega_x \to \infty$, is always stable (see Section 16). The conditions of stability, cited at the beginning of the paragraph, reflect the basic characteristics of the aircraft motion, however they do need refinement. Even from the approximate analysis it was clear that the inertial moments, arising during rotations of the aircraft, can change qualitatively the characteristics of the motion, in particular they can make it unstable.

Let us look in greater detail at the stability of aircraft motion during rotation relative to the longitudinal axis OX_1 at a constant angular rolling velocity ω_x = const. In this case, as the original motion, we can look at the straight and steady flight with a small angle of slope of the path to the horizon. We can assume that in the condition of steady rotation at an angular velocity $\omega_x = \Omega$ = const, the flying speed remains constant, i.e., V_0 = const; in the same manner the equation of the projections of forces for the axis OX_1 is automatically satisfied and in the future we shall not look at this equation.

To study the stability "in the small", let us transform the equation (1.28) into equations in variations; after linearizing them relative to a certain steady motion, the characteristics of which will be the presence of the angular rolling velocity $\omega_x = \Omega$, we designate the parameters of this system by the index zero. With small deviations from the steady state system the parameters of motion can be written in the following form:

$$\begin{aligned}
\alpha &= \alpha_0 + \Delta\alpha; \\
\bar{\omega}_z &= \bar{\omega}_{z0} + \Delta\bar{\omega}_z; \\
\beta &= \beta_0 + \Delta\beta; \\
\bar{\omega}_y &= \bar{\omega}_{y0} + \Delta\bar{\omega}_y; \\
\bar{\omega}_x &= \Omega = \text{const}; \\
\vartheta &= \Delta\vartheta; \\
\gamma_0 &= \Omega \cdot t.
\end{aligned}\qquad(2.5)$$

If we substitute the values of expressions (2.5) into the basic system of equations (1.28), we find in each equation three groups of terms. One group will consist of the products of small values - these terms can be omitted with linearization as terms having a second order of smallness. The second group of terms in the equations will contain values corresponding to the original system of steady rotation - these terms in total must be equal to zero since the original conditions also satisfy the system of equations of motion. And finally the third group of terms of each equation will represent terms containing the desired variations of parameters of first order. The system of equations for the original system of steady rotation has in this case the following form:

$$\begin{aligned}
\alpha_0' &= -\frac{c_y^\alpha}{2}\alpha_0 + \mu\bar{\omega}_{z0} - \mu\Omega\beta_0 + \frac{2g_0\tau_m^2}{l\mu}\cos\gamma_0; \\
\bar{\omega}_{z0}' &= -\mu A\Omega\bar{\omega}_{y0} + \bar{m}_{zb}^\alpha\alpha_0 + \bar{m}_{zb}^{\omega_z}\cdot\bar{\omega}_{z0} + \bar{m}_{zb}^{\delta e}\cdot\delta_e; \\
\beta_0' &= \mu\bar{\omega}_{y0} + \mu\Omega(\alpha_0 + \alpha_m) + \frac{c_z^\beta}{2}\beta_0 + \frac{c_z^{\delta r}}{2}\delta_r + \frac{2g_0\tau_m^2}{l\cdot\mu}\sin\gamma_0; \\
\bar{\omega}_{y0}' &= B\mu\Omega\bar{\omega}_{z0} + \bar{m}_y^\beta\beta_0 + \bar{m}_y^{\omega_y}\cdot\bar{\omega}_{y0} + \bar{m}_y^{\delta r}\cdot\delta_r; \\
\bar{\omega}_{x0}' &= \bar{m}_x^{\omega_x}\cdot\Omega + \bar{m}_x^{\delta a}\cdot\delta_a; \\
\gamma_0' &= \mu\Omega.
\end{aligned}\qquad(2.6)$$

Let us note that during flight with large dynamic heads and at high speeds, as calculations show, the influence of gravitational forces on the motion of an aircraft relative to the center of mass in the types of maneuvers studied here is negligibly small and the gravitational terms in the equations of motion can be omitted. Then in analyzing the stability of motion of an aircraft (without taking the gravitational terms into account) the requirement for studying the kinematic equation no longer exists and in final form /44 the system of equations in the variations, after excluding equations (2.6) for the steady state system and neglecting the terms of second order of smallness, will have the form:

$$\begin{aligned}
\alpha' &= -\frac{c_y^\alpha}{2}\,\alpha + \mu\overline{\omega}_z - \mu\Omega\beta; \\[4pt]
\overline{\omega}_z' &= \overline{m}_z^{\overline{\omega}_z}\cdot\overline{\omega}_z + \overline{m}_{z\mathrm{b}}^\alpha\alpha - A\,\mu\Omega\overline{\omega}_y; \\[4pt]
\beta' &= \frac{c_z^\beta}{2}\,\beta + \mu\overline{\omega}_y + \mu\Omega\alpha; \\[4pt]
\overline{\omega}_y' &= \overline{m}_y^\beta\cdot\beta + \overline{m}_y^{\overline{\omega}_y}\cdot\overline{\omega}_y + B\,\mu\Omega\overline{\omega}_z.
\end{aligned} \qquad (2.7)$$

The variability in the flying speed can be taken into account in equations (2.7) through changes in the aerodynamic coefficients of damping, as noted in Section 4. We must note that the equation of the moments acting on the aircraft relative to the axis OX_1, is automatically satisfied under the condition

$$m_x^\beta = m_x^{\overline{\omega}_y} = C = 0. \qquad (2.8)$$

In other cases when expression (2.8) is not satisfied, the condition $\omega_x = \Omega = $ const requires additional deflections of the lateral controls (δ_a), since in the condition of disturbed motion the angle of side slip β and the angular yawing velocity ω_y will be changed.

Let us note the following quite interesting result which follows from the calculations carried out. The values of the parameters of the original flight path and the terms from the gravitational forces were not included in the system of equations in variations (2.7). This can be explained by the fact that in satisfying conditions (2.8) the equation for the rolling motion is separated and can be solved separately. As a result the system of equations (1.26) becomes a system of linear equations with a periodic disturbing function, caused by the gravitational forces, and in linearization the terms from these distrubances remain only in the equations of the original motion. Thus, regardless of the type of controlled motion of the aircraft, described by solving the system of equations (2.6), if expressions (2.8) are not satisfied, its stability is determined by the system of linear equations (2.7). When $\Omega = $ const, the system of equations (2.7) is a system of linear equations with constant coefficients and when $\Omega(t)$ it is a variable value depending on time; the system of equations is a system of linear equations with variable coefficients.

Let us transform equations (2.7) for the case $\Omega = $ const into a system of two equations of first order for the variables α and β. After differentiating the first and third equations of system (2.7) /45 and satisfying the elementary transformations, we find the equations of motion of an aircraft in the form

$$\alpha'' + \left(-\overline{m}_{z\mathrm{b}}^{\overline{\omega}z} + \frac{c_y^\alpha}{2}\right)\alpha' + \left(-\mu\overline{m}_{z\mathrm{b}}^\alpha - \frac{\overline{m}_{z\mathrm{b}}^{\overline{\omega}z}\cdot c_y^\alpha}{2} - A\mu^2\Omega^2\right)\alpha +$$

$$+ (1+A)\,\Omega\mu\beta' - \left(+\overline{m}_{z\mathrm{b}}^{\overline{\omega}z} + A\,\frac{c_z^\beta}{2}\right)\mu\Omega\beta = 0;$$

$$\beta'' + \left(-\overline{m}_y^{\overline{\omega}y} - \frac{c_z^\beta}{2}\right)\beta' + \left(-\mu\overline{m}_y^\beta + \frac{\overline{m}_y^{\overline{\omega}y}\cdot c_z^\beta}{2} - B\mu^2\Omega^2\right)\beta -$$

$$- (1+B)\,\Omega\mu\alpha' + \left(\overline{m}_y^{\overline{\omega}y} - B\,\frac{c_y^\alpha}{2}\right)\mu\Omega\alpha = 0.$$

$$(2.9)$$

From comparison of equations (2.9) with (1.36) and (1.43) it
is clear that the rotation of an aircraft at a constant angular
rolling velocity Ω = const has led to an interrelationship between
the motions involving the angle of attack and those involving yaw-
ing, the degree of which increases in proportion to the square of
the angular rolling velocity Ω. In connection with this, at small
angular rolling velocities the equations can be separated with an
accuracy up to values of the second order of smallness.

Let us cite the conditions for the stability of solving the
system of equations (2.7) for the case of a constant angular roll-
ing velocity $\Omega = \Omega_0$ = const. If we make the necessary computations,
we find the expression for the characteristic equation of the system
of equations of motion (2.7) in the following form:

$$\lambda^4 + A_3\lambda^3 + A_2\lambda^2 + A_1\lambda + A_0 = 0, \qquad (2.10)$$

where the coefficients A_3, A_2, A_1 and A_0 are functions of the par-
ameters of motion and the aerodynamic characteristics of the air-
craft:

$$A_3 \cdot \frac{c_y^\alpha}{2} - \overline{m}_{z\mathrm{b}}^{\overline{\omega}z} - \frac{c_z^\beta}{2} - \overline{m}_y^{\overline{\omega}y}; \qquad (2.11)$$

$$A_2 = \left(\overline{m}_{z\mathrm{b}}^{\omega z} + \overline{m}_y^{\overline{\omega}y}\right)\left(\frac{c_z^\beta}{2} - \frac{c_y^\alpha}{2}\right) + \overline{m}_{z\mathrm{b}}^{\omega z}\cdot\overline{m}_y^{\overline{\omega}y} - \frac{c_z^\beta}{2}\cdot\frac{c_y^\alpha}{2} - \qquad (2.12)$$

$$- \mu\left(\overline{m}_y^\beta + \overline{m}_{z\mathrm{b}}^\alpha\right) + \mu^2\Omega_0^2(1+AB);$$

$$A_1 = \left(\mu\overline{m}_y^\beta - \frac{c_z^\beta}{2}\overline{m}_y^{\overline{\omega}y}\right)\left(\overline{m}_{z\mathrm{b}}^{\overline{\omega}z} - \frac{c_y^\alpha}{2}\right) - \mu^2\Omega_0^2\left(\overline{m}_{z\mathrm{b}}^{\overline{\omega}z} + \overline{m}_y^{\overline{\omega}y}\right) + \qquad (2.13)$$

$$+ \left(\mu\overline{m}_{z\mathrm{b}}^\alpha + \frac{c_y^\alpha}{2}\overline{m}_{z\mathrm{b}}^{\overline{\omega}z}\right)\left(\overline{m}_y^{\overline{\omega}y} + \frac{c_z^\beta}{2}\right);$$

$$\frac{A_0}{\mu^2} = \left[\left(-\overline{m}_{z\,b}^{\alpha} - \frac{c_y^{\alpha}\,\overline{m}_z^{\overline{\omega}_z}}{2\mu} \right) - A\Omega^2\mu \right] \left[\left(-\overline{m}_y^{\beta} + \frac{c_z^{\beta}\,\overline{m}_y^{\overline{\omega}_y}}{2\mu} \right) - B\Omega^2\mu \right]\mu^2 +$$

$$+ \Omega^2 \left(-\overline{m}_{z,b}^{\overline{\omega}_z} - A\frac{c_z^{\beta}}{2} \right) \left(-\overline{m}_y^{\overline{\omega}_y} + B\frac{c_y^{\alpha}}{2} \right)\mu^2. \tag{2.14} \quad \underline{/46}$$

For the stability of motion it is necessary that all real parts of the roots of the characteristic equation (2.10) be negative. On the basis of the Ross-Hurwitz criteria, we can write the conditions of stability of motion in the form

$$A_3 \geqslant 0; \quad A_2 \geqslant 0; \quad A_0 \geqslant 0; \\ R = A_3(A_2 A_1 - A_3 A_0) - A_1^2 \geqslant 0. \Big\} \tag{2.15}$$

From expressions (2.11) and (2.12) it follows that for the stable motion of an aircraft the conditions $A_3 > 0$ and $A_2 > 0$ must be satisfied for all values of the angular rolling velocity. Calculations also show that in this case the condition $R > 0$ is also satisfied for all possible values of Ω. There remains the single condition of stability $A_0 > 0$, which as a function of the value Ω can either be satisfied or not satisfied. This condition, since it is associated with the sign of the free term, is a condition of aperiodic stability of aircraft motion during rotation at a constant angular velocity relative to the longitudinal axis and will be analyzed in detail below. When $A_0 \leq 0$ the unstable motion of an aircraft has an aperiodic character.

8. Analysis of the Condition of Aperiodic Stability of Motion During Steady Rolling Turn

Let us proceed to an analysis of the free term of the characteristic equation A_0. The basic parameters, which determine the aperiodic stability of motion of an aircraft during isolated lateral and longitudinal motions, are the values of the excess static stability $\overline{m}_{z\,b}^{\alpha}$ and $\overline{m}_y^{\beta}$. As noted in Section 5, the condition of stability of isolated longitudinal and lateral motion is the satisfaction of the inequality

$$\left(-\overline{m}_{z\,b}^{\alpha} - \frac{c_y^{\alpha}\,\overline{m}_z^{\overline{\omega}_z} b}{2\mu} \right) \geqslant 0; \\ \left(-\overline{m}_y^{\beta} + \frac{c_z^{\beta}\,\overline{m}_y^{\overline{\omega}_y}}{2\mu} \right) \geqslant 0. \Bigg\} \tag{2.16}$$

With spatial motion at a high value of the angular rolling $\underline{/47}$ velocity $\omega_x = \Omega_0$, the condition of stability is complicated and in expanded form is written in the following manner:

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$$\left[\left(-\overline{m}_{z\,b}^{\alpha}-\frac{c_{y}^{\alpha}\,\overline{m}_{z}^{\overline{\omega}_{z}}}{2\mu}\right)-A\Omega_{0}^{2}\mu\right]\left[\left(-\overline{m}_{y}^{\beta}+\frac{c_{z}^{\beta}\,\overline{m}_{y}^{\overline{\omega}_{y}}}{2\mu}\right)-B\Omega_{0}^{2}\mu\right]+$$

$$+\Omega_{0}^{2}\left(-A\frac{c_{z}^{\beta}}{2}-\overline{m}_{z\,b}^{\overline{\omega}_{z}}\right)\left(B\frac{c_{y}^{\alpha}}{2}-\overline{m}_{y}^{\overline{\omega}_{y}}\right)\geqslant0. \qquad (2.17)$$

Direct proof will show that when $\Omega_0 \to 0$, conditions (2.16) follow from inequality (2.17). The condition of stability (2.7) given above is the basic one. The purpose of further studies in this paragraph will be determining the types of stability regions as a function of the different parameters of an aircraft and finding approximate criteria of stability.

As follows from expression (2.17) and from the approximate qualitative discussions, given at the beginning of the chapter, the basic factor which determines the stability of motion of an aircraft is the relationship between the excess static stability $\overline{m}_y^{\beta}$ and $\overline{m}_{z\,b}^{\alpha}$ and the values which are proportional to the square of the angular rotational velocity of the aircraft relative to the longitudinal axis. In this respect in first order let us determine the form of the boundaries of the stability regions of aircraft motion on a plane with the coordinates $\overline{m}_{z\,b}^{\alpha}$ and $\overline{m}_y^{\beta}$. As follows from condition (2.17), the boundaries of the stability region in these coordinates is a hyperbola described by the relationship

$$X \cdot Y = -K, \qquad (2.18)$$

where

$$\left.\begin{aligned}
\dot{X}&=X_{1}-A\mu\Omega_{0}^{2}; \quad X_{1}=\left(-\overline{m}_{z\,b}^{\alpha}-\frac{c_{y}^{\alpha}\,\overline{m}_{z}^{\overline{\omega}_{z}}}{2\mu}\right); \\
Y&=Y_{1}-B\mu\Omega_{0}^{2}; \quad Y_{1}=\left(-\overline{m}_{y}^{\beta}+\frac{c_{z}^{\beta}\,\overline{m}_{y}^{\overline{\omega}_{y}}}{2\mu}\right); \\
K&=K_{0}\Omega_{0}^{2}; \quad K_{0}=\left(-A\frac{c_{z}^{\beta}}{2}-\overline{m}_{z\,b}^{\overline{\omega}_{z}}\right)\left(B\frac{c_{y}^{\alpha}}{2}-\overline{m}_{y}^{\overline{\omega}_{y}}\right).
\end{aligned}\right\} \qquad (2.19)$$

Examples of the boundaries of the stability regions for different values of the angular rolling velocity Ω_0 are given on Figure 2.3. From expressions (2.18) and (2.19) it follows that the hyperbolas in the coordinate axes $(\overline{m}_{z\,b}^{\alpha}, 0, \overline{m}_y^{\beta})$ have asymptotes described by the equations:

$$-\overline{m}_{z\,b}^{\alpha}=A\mu\Omega_{0}^{2}+\frac{c_{y}^{\alpha}\,m_{z\,b}^{\omega_{z}}}{2\mu}\simeq A\mu\Omega_{0}^{2}; \qquad (2.20)$$

$$-\overline{m}_{y}^{\beta}=B\mu\Omega_{0}^{2}-\frac{c_{z}^{\beta}\,\overline{m}_{y}^{\omega_{y}}}{2\mu}\simeq B\mu\Omega_{0}^{2}. \qquad (2.21)$$

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In those regions of the plane where the values $\overline{m}^\alpha_{z\mathrm{b}}$ and $\overline{m}^\beta_y$ differ substantially from one another, the hyperbolas approach the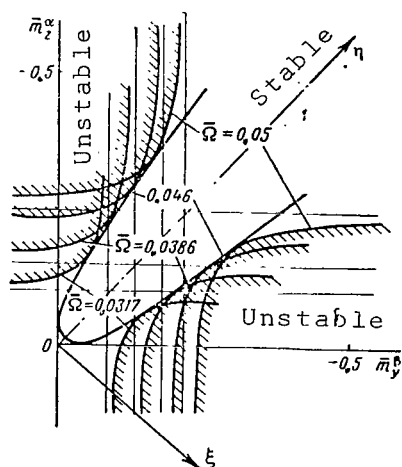ir asymptotes (see Fig. 2.3). Hence, in particular, it follows that in these regions of the parameter values of static stability of an aircraft approximate criteria of stability can be used, by substituting the equation of the hyperbola with the equations of its asymptotes. On the basis of these discussions we can find approximate expressions for the angular rolling velocities, in which the losses in stability occur. Using the equations of the asymptotes (2.20) and (2.21) we find

Fig. 2.3.

$$\overline{\omega}_\alpha = \sqrt{\frac{-\overline{m}^\alpha_{z\mathrm{b}}}{A\mu}}\,; \qquad (2.22)$$

$$\overline{\omega}_\beta = \sqrt{\frac{-\overline{m}^\beta_y}{B\mu}}\,. \qquad (2.23)$$

Formulas (2.22) and (2.23) can be feasibly used in making a preliminary analysis in those widely used cases when the excess longitudinal and lateral stabilities of an aircraft differ substantially, and the damping of the motion is not too high. From expressions (2.19) it follows that the centers of the hyperbolas lie on a straight line, described by the equation

$$\frac{X_1}{A} = \frac{Y_1}{B}\,. \qquad (2.24)$$

Since the values $\dfrac{c^\alpha_y \overline{m}^{\omega_z}_{z\mathrm{b}}}{2\mu}$ and $\dfrac{c^\beta_z \overline{m}^{\omega_y}_y}{2\mu}$ are usually substantially smaller than the values of $\overline{m}^\alpha_{z\mathrm{b}}$ and $\overline{m}^\beta_y$, then the equation of the lines of the centers can be approximately rewritten in the form

$$\overline{m}^\alpha_{z\mathrm{b}} = \overline{m}^\beta_y \frac{A}{B} = \overline{m}^\beta_y \frac{J_y - J_x}{J_z - J_x} \cdot \left(\frac{J_y}{J_z}\right), \qquad (2.25)$$

i.e., the slope of the lines of the centers of the hyperbolas prac- <u>/49</u>
tically do not depend on the flight path, but are determined by the inertial characteristics of the aircraft.

With a change in the value of the angular rolling velocity of the aircraft the boundaries of the regions of stability of motion are shifted (see Fig. 2.3). Let us introduce the equation of the enveloping boundaries of stability in the region of variation Ω_0 from zero to ∞. Construction of such an envelope permits clarifying the region of the "absolute" stability of motion of an aircraft for the entire possible range of the angular rolling velocities.

For the envelope the following system of equations must be satisfied:

$$F(\Omega^2) = A_0(\Omega^2) = XY + K = 0;$$
$$\frac{\partial F(\Omega^2)}{\partial \Omega^2} = 0. \qquad (2.26)$$

Excluding the parameter Ω^2 from equations (2.26) we find the equation of the envelope

$$-K_0^2 + 2K_0(A\mu Y_1 + B\mu X_1) = (B\mu X_1 - A\mu Y_1)^2. \qquad (2.27)$$

In order to reduce the equation of the envelope (2.26) to a convenient form, let us introduce new variables η and ξ using relationships

$$\eta = A\mu Y_1 + B\mu X_1;$$
$$\xi = B\mu X_1 - A\mu Y_1. \qquad (2.28)$$

It is easy to prove that the axis $O\eta$ agrees with the line of the centers of the hyperbolas. In fact, if we equate ξ to zero, then we find the equation of the axis $O\eta$, which agrees with the equation of the line of the centers of the hyperbolas (2.24).

The system of axes $\xi O\eta$ is scalene and the axes $O\xi$ and $O\eta$ form identical angles with the axis OX_1 (Fig. 2.4), the value of which is equal to

$$|\varphi_1| = \text{arctg} \frac{B}{A}. \qquad (2.29)$$

Making a substitution of variables in equation (2.27), according to formulas (2.28), we find the equation of the envelope in the system of axes $\xi O\eta$:

$$2K_0\left(\eta - \frac{K_0}{2}\right) = \xi^2. \qquad (2.30)$$

From expression (2.30) it follows that the envelope is a parabola, whose axis coincides with the line of the hyperbola centers (see Fig. 2.4).

Let us show that the parabola of expression (2.30) is tangent to the axes OX_1 and OY_1. The values ξ and η, when $X_1 = 0$ or $Y_1 = 0$, are mutually related by the relationship

$$\eta = \pm \xi. \qquad (2.31)$$

If we substitute equation (2.3) into the equation of the parabola (2.30), we find that the equation can be satisfied and the tangency occurs at values of ξ and η determined from the equations

$$\xi = K_0;$$
$$\eta = K_0. \qquad (2.32)$$
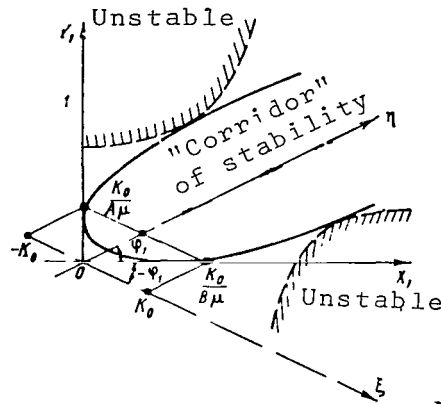
Fig. 2.4.
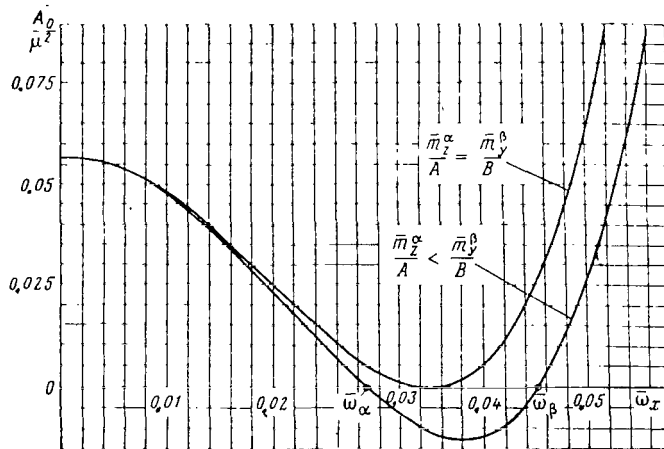
Fig. 2.5.

From expression (2.31) it follows that the envelope is tangent to the axes $X_1 O Y_1$ at the points with coordinates on the axis OY_1 equal to $K_0/A\mu$ and on the axis OX_1 equal to $(K_0/B\mu)$.

The equation of the envelope defines on the plane $(\overline{m}^\alpha_{zb}, 0, \overline{m}^\beta_y)$ a certain "corridor". In choosing the values $\overline{m}^\alpha_{zb}$ and $\overline{m}^\beta_y$ from this corridor the aircraft will be stable at any value of the angular rolling velocity. It is obvious that the wider the corridor the easier will be the choice of parameters such as $\overline{m}^\alpha_{zb}$ and $\overline{m}^\beta_y$. Quantitatively we can determine the width of the corridor by analyzing the value $|\xi|$ when η = const. From expression (2.30) it follows that

$$|\xi| \sim \sqrt{2K_0}.$$

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Taking into account the relationship between the scales of the variables in the axes $\xi O \eta$ and $X_1 O Y_1$ of expression (2.28) we find that the width of the corridor in the system of axes $X_1 O Y_1$ is proportional to the expression

$$\frac{1}{\mu}\sqrt{\left(-A\,\frac{c_z^\beta}{2}-\overline{m}_{z\mathrm{b}}^{\overline{\omega}z}\right)\left(B\,\frac{c_y^\alpha}{2}-\overline{m}_y^{\overline{\omega}y}\right)}. \qquad (2.33)$$

The width of the corridor is increased with increase in the values of the coefficients of damping and is decreased with increase in the height of flight of the aircraft. With zero values of the coefficients of damping c_y^α, c_z^β, $\overline{m}_y^{\overline{\omega}y}$, $\overline{m}_z^{\overline{\omega}z}$ the corridor is degenerated into a line and the hyperbolas are merged with their asymptotes.

Construction of the regions of stability in the plane of the parameters $\overline{m}_y^\beta$ and $\overline{m}_{z\mathrm{b}}^\alpha$ is convenient in analyzing stability, when the parameters of an aircraft are either not determined or may be varied in a certain range. In those cases when the aerodynamic parameters of an aircraft are a priori determined, for analysis of stability it is convenient to simply construct a graph of the function $A_0(\Omega)$ and use conditions (2.15). On such a graph we can see with special clarity the meaning of the critical angular rolling velocities introduced above. Since A and B are positive numbers, then for a statically stable aircraft the following relationships are always satisfied;

$$A_0(0)>0; \quad A_0(\Omega\to\infty)>0. \qquad (2.34)$$

On the other hand $A_0(\Omega)$ is a Ω^2 parabola which, as follows from expressions (2.34), can have either two or no zeros, or one zero in the special case of tangency of the curve $A_0(\Omega)$ of the abscissa axis. As an example, on Figures 2.5 and 2.6 are plotted the functions $A_0(\Omega)$ for different relationships between $\overline{m}_{z\mathrm{b}}^\alpha$ and $\overline{m}_y^\beta$ and different /52 coefficients of damping. From these figures it is clear that the critical angular rolling velocities $\overline{\omega}_\alpha$, $\overline{\omega}_\beta$ which were introduced above using equations (2.22) and (2.23) correspond to zeros of the function $A_0(\Omega)$, in which all terms of damping are taken equal to zero. With real values of the coefficients of damping, when the aircraft has no oscillation dampers, the approximate values of the critical angular velocities, determined from formulas (2.22) and (2.23) are near to the zeros of the function $A_0(\Omega)$.

The expression for the zeros of the function $A_0(\Omega)$ can be obtained precisely in explicit form. For this purpose, if we analyze Ω_0^2 as the sought parameter, we will transform the expression for $A_0(\Omega_0^2)$ into the form

$$(\Omega_0^2)^2-\left\{\left[\frac{X_1}{A\mu}+\frac{Y_1}{B\mu}\right]-\frac{K_0}{AB\mu^2}\right\}\Omega_0^2+\frac{X_1 Y_1}{AB\mu^2}=C, \qquad (2.35)$$

whence it follows that

$$\Omega_0^2 = \frac{1}{2}\left\{\left[\frac{X_1}{A\mu} + \frac{Y_1}{B\mu}\right] - \frac{K_0}{AB\mu^2}\right\}$$

$$\pm \frac{1}{2}\sqrt{\left\{\left[\frac{X_1}{A\mu} + \frac{Y_1}{B\mu}\right] - \frac{K_0}{AB\mu^2}\right\}^2 - \frac{4X_1Y_1}{AB\mu^2}}. \qquad (2.36)$$

For the case of small damping with "dispersed" values of the critical rolling velocities, from formula (2.36) we can find the refined approximate expression for the critical rolling velocities in a form which is convenient for computation.
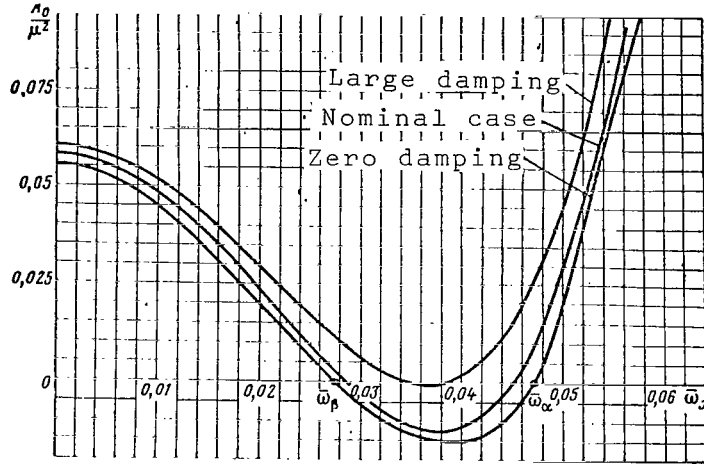
Fig. 2.6.

Using the definitions introduced above, we can write /53

$$\overline{\omega}_\alpha = \frac{X_1}{A\mu};$$

$$\overline{\omega}_\beta = \frac{Y_1}{B\mu}. \qquad (2.37)$$

If we neglect the small term $(K_0/AB\mu^2)^2$ under the radical and retain only the first term of the representation of the square root in the form of a Taylor series, we find

$$\overline{\omega}_\alpha^* \simeq \overline{\omega}_\alpha\left[1 - \frac{K_0}{2AB\mu^2}\cdot\frac{1}{(\overline{\omega}_\alpha^2 - \overline{\omega}_\beta^2)}\right]; \qquad (2.38)$$

$$\overline{\omega}_\beta^* \simeq \overline{\omega}_\beta\left[1 + \frac{K_0}{2AB\mu^2}\cdot\frac{1}{(\overline{\omega}_\alpha^2 - \overline{\omega}_\beta^2)}\right]. \qquad (2.39)$$

Formulas (2.38) and (2.39), taking into account the comments made above, agree well with the calculation. From these formulas, in

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particular it is clear that the damping is near the zeros of the function $A_0(\Omega)$. For example, in the case when $\bar\omega_\alpha > \bar\omega_\beta$, from expressions (2.38) and (2.39) it follows that

$$\left.\begin{array}{l} \bar\omega_\beta < \bar\omega_\beta^*; \\ \bar\omega_\alpha > \bar\omega_\alpha^*. \end{array}\right\} \tag{2.40}$$

In the presence of oscillation dampers on the aircraft, and also in those cases when the values $\bar m_{zb}^\alpha$ and $\bar m_y^\beta$ are near, the function $A_0(\Omega)$ in general may not have zeros. Let us determine by what conditions the parameters of the aircraft will be satisfied in order that this be done. In order for the function $A_0(\Omega)$ not to have zeros, it is necessary and sufficient that it be positive with a value of the angular rolling velocity Ω, when the derivative $(\partial A_0/\partial\Omega)$ is equal to zero.

The boundary of such a region of parameters is determined from the condition of simultaneous conversion to zero of the expressions

$$A_0(\Omega^2) = 0;$$
$$\frac{\partial A_0(\Omega^2)}{\partial \Omega^2} = 0:$$

Comparing these conditions with expressions (2.26), we find that the equation for such a boundary is the equation of the envelope introduced above [see equation (2.7)]. However, as has been mentioned above, the values of the damping are usually small and the conditions of stability of an aircraft may be written in the form of an inequality, from which it follows that the motion is stable in those cases when the angular rolling velocity lies outside the /54 zone of critical angular velocities:

$$\Omega \leqslant \min(\bar\omega_\alpha^*, \bar\omega_\beta^*), \tag{2.41}$$

or

$$\Omega \geqslant \max(\bar\omega_\alpha^*, \bar\omega_\beta^*).$$

In equation (2.41) the sign min denotes the smaller of the two numbers $\bar\omega_\alpha^*$, $\bar\omega_\beta^*$, and the sign max denotes the larger of them; these are the zero functions $A_0(\Omega)$.

The appearance of stability losses of motion by a turning aircraft (which has been analyzed in detail) for a certain critical angular rolling velocity is not unique. In mechanics similar phenomena are known. In first order in this respect it follows to note the phenomenon of resonance of rotating shafts (see for example [13]). The phenomenon of resonance involves the development of resonance oscillations which may lead to disruption of the shaft when the angular velocity of rotation of the shaft becomes equal to the natural frequency of the lateral oscillations. An analogous

phenomenon may be observed in the operation of a centrifuge in a washing machine. Resonance in this latter case is observed in the form of two pulses: when the centrifuge is started up and when it is slowed down. The resonance is stronger as the displacement of the center of gravity of the centrifuge relative to the axis of rotation becomes greater.
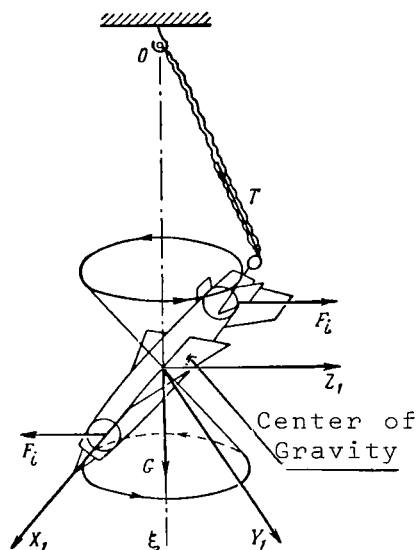
A good illustration of the phenomena noted above may be an experiment with a model of an aircraft. The model of the aircraft, whose basic mass is concentrated in the fuselage, is suspended on a rubber band, so that the weight of the model is compensated by the tensile strength of the rubber. With a deflection by the angles α and β the model will cause oscillations relative to the center of mass acted on by the reduced force T, equal to the weight of the model (Fig. 2.7). Let us twist the rubber band and drop the model. With a weak twist of the elastic the model of the aircraft will revolve slowly relative to the major inertial axis OX_1 with little deviation from the vertical. If the rubber is first wound tightly, the angular rotational velocity of the model will be great and will reach a critical value, in which case the spatial angle /55 of attack begins to increase and the model, by rotating, "describes" a cone in space, i.e., seeming losses in motion stability are observed "in the small" (see Fig. 2.7).

Fig. 2.7.

Such a motion of the model is similar to the motion of an aircraft with steady rolling turn, the only difference being that the degree of stability in the model with motion relative to the axes OY_1 and OZ_1 is identical and in the aircraft it is usually different.

Let us note that the angular rotational velocity of the model, after reaching the critical value, practically does not increase, since the change in the spatial angle of attack of the model in the region of the critical velocity acts similarly to the centrifugal regulator (Watt regulator). With an increase in the angular velocity the angle of attack increases, thus leading to an increase in the inertial moment of the load on the rubber relative to the axis $O\xi$ and correspondingly retards the development of angular rotational velocity.

9.   Stability of Motion During Steady Rolling Turn in Several Extreme Cases.

Let us look at several simplified extreme cases of the motion of an aircraft turning in a roll. First let us look at the case of aircraft motion, turning relative to the longitudinal axis on which no aerodynamic forces and moments are acting, which case may be valid during flight with a very small dynamic head ($q \to 0$). If in this case we further assume (for simplification of all calculations) that the ellipsoid of inertia has an axis of symmetry, i.e., $J_y = J_z = 1$, then the equation of motion of the aircraft will have the form:*

$$
\left.
\begin{aligned}
J \frac{d\omega_z}{dt} &= -(J - J_x)\,\omega_x \omega_y; \\
J \frac{d\omega_y}{dt} &= (J - J_x)\,\omega_x \omega_z; \\
J_x \frac{d\omega_x}{dt} &= 0.
\end{aligned}
\right\}
\qquad (2.42)
$$

From Equations (2.42) we find the equations for change in the angular velocities ω_z and ω_y:

$$
\left.
\begin{aligned}
\ddot{\omega}_z &= -\left(\frac{J - J_x}{J}\,\omega_x\right)^2 \omega_z; \\
\ddot{\omega}_y &= -\left(\frac{J - J_x}{J}\,\omega_x\right)^2 \omega_y; \quad \omega_x = \text{const.}
\end{aligned}
\right\}
\qquad (2.43)
$$

Solution to Equations (2.43), when $J > J_x$, is always stable and can /56 be written in the form

$$
\left.
\begin{aligned}
\omega_z &= a \cos(\eta t); \\
\omega_y &= a \sin(\eta t),
\end{aligned}
\right\}
\qquad (2.44)
$$

where

$$
\eta = \frac{J - J_x}{J}\,\omega_x;
$$

$$
a = \omega_z(0) = \text{const}; \quad \omega_y(0) = 0.
$$

From expressions (2.44), assuming that $\omega_x = \text{const}$, it is easy to see that the total value of the angular velocity for the entire

---

*This case is studied in classical courses of theoretical mechanics (see [9-10]).

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time of motion remains constant ($\omega_z^2 + \omega_y^2 = a^2$), but the direction of the vector of angular velocity will vary relative to the body (precess) at a frequency η. Hence, it especially follows that the steady rotation of an aircraft without precession is possible only relative to the major inertial axis ($a = 0$), and rotation relative to any other axis will lead to precession.

If all inertial moments of the aircraft are different (the Euler case) then it is possible to show (see [9], [10]) that its rotation relative to the axis with minimal and maximal inertial moments is stable. In such case the rotation relative to the major inertial axis OX_1, usually corresponding to the minimal inertial moment with $\omega_x = $ const, will occur without precession. In the general case the motion of an aircraft is described by elliptical functions.

Usually for aircraft the relationship $J_y \sim J_z$ is satisfied, therefore, the motion in these assumptions will be almost periodic and near the motion of a symmetric body studied above.

Let us look at the second extreme case when the moments of the static stability of an aircraft m_y^β and m_z^α are high, and the inertial moments are quite small. This case approximately corresponds to the flight of an aircraft in a system with large dynamic heads and a small angular rolling velocity, when the relative role of the aerodynamic forces is high in comparison with the inertial forces. Then when $\omega_x = $ const, during the time of the motion, approximate relationships such as these will be satisfied:

$$\alpha = \alpha_0 = \text{const};$$
$$\beta = 0,$$

i.e., steady rotation of the aircraft will occur relative to the velocity vector of the flight.

Let us look at this latter extreme case of motion, when the aircraft has infinitely large inertia, i.e., $J_y = J_z \to \infty$, the moments of static stability are small and $\omega_y(0) = \omega_z(0) = 0$. In this case the steady rotation of an aircraft will occur relative to the major inertial axis, i.e., practically relative to the longi-
tudinal axis of the fuselage. If at the initial moment of time when $\gamma = 0$, the relationship $\alpha = \alpha_0$ and $\beta = 0$ is satisfied, then when $\gamma = \pi/2$, we find $\alpha = 0$ and $\beta = \alpha_0$. Consequently, the angle of attack and angle of side slip will be periodic functions of the angle of bank (see Fig. 1.6): $\alpha = \alpha_0 \cos \gamma$; $\beta = \alpha_0 \sin$.

It is natural that in the general case the motion of an aircraft has a more complex character, especially when the values of the aerodynamic and inertial moments are of one order. However, in the general case we can almost always observe elements of the extreme cases analyzed above.

10. The Dynamic Characteristics of Modern Aircraft

Types of interactions between longitudinal and lateral motions have always existed, however they have become quite substantial only with the appearance of supersonic aircraft. Which of these changes in the classification of aircraft and their aerodynamic characteristics have led to such results? Let us look briefly at the basic tendencies of the changes in the characteristics of maneuvering aircraft from the viewpoint of interaction between the longitudinal and lateral motions.
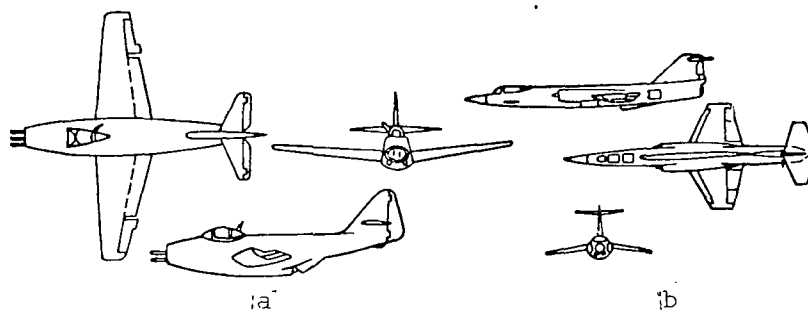

Fig. 2.8.

High flying speeds have led to substantial change in the forms of the specific elements of aircraft. Supersonic aircraft (Fig. 2.8,b) in comparison with subsonic (see Fig. 2.8,a) have a more elongated shape to the fuselage, narrower wings of smaller length which cause the designers to distribute the basic loads in the fuselage. As a result, the ratio of the moments of inertia J_z/J_x, J_y/J_x, which was of the order of 2 - 3 for subsonic aircraft, approaches a value of the order of 10 - 15 for supersonic aircraft.

From formulas (2.22) and (2.23) it follows that with a constant value of the moment of inertia $J_z(J_x)$ and the excess directional /58 and longitudinal stability, such a change in the ratios between the moments of inertia of the aircraft relative to the longitudinal and lateral axes leads to a decrease in the critical rolling velocities by almost 1.5 - 2 times.

For maneuvering aircraft of contemporary groupings it is characteristic that the excess longitudinal stability during transition from subsonic to supersonic speeds increases substantially, which can be explained by the backward shift along the flow of the wing focal point, i.e., by the increase in the derivative m_{zy}^c (Fig. 2.9). On the other hand, the excess directional stability of the aircraft, as noted above, with an increase in the M number is substantially decreased (Fig. 2.10). All this leads to the fact that, as a rule, at subsonic flying speeds the least critical rolling velocity is ω_α, i.e., the critical velocity determined by the pitching motion, and with transition to supersonic numbers M(M > 1),

ω_α and ω_β swap places and the least critical rolling velocity becomes the value ω_β, i.e., the critical velocity determined by the yawing motion (Fig. 2.11). Along with the above, in supersonic aircraft the value of the maximal proposed rolling velocity, as a rule, is greater thus causing a slight decrease in damping during the transition involving wings of small length. Summing up, it follows to note that all the factors mentioned above decrease the critical rolling velocities and make it easier to reach them in flight. Typical dependences of the critical rolling velocities on flying speed for aircraft of various types are illustrated also on Figure 2.12.
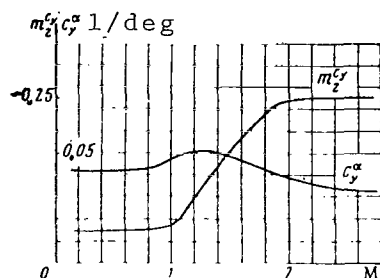
Fig. 2.9
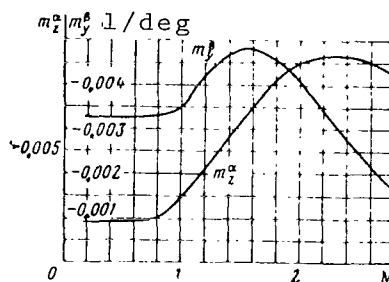
Fig. 2.10

The results found in the preceding paragraphs, regardless of the simplified formulation of the problem, permit finding several qualitative representations of the general case of spatial motion of an aircraft, accompanied by strong rolling. In fact the relationships cited above indicate that if the angular rolling velocity in the process of motion approaches or exceeds the least of the critical rolling velocities then the stability of the aircraft motion is either decreased or in general may be disturbed; in such case the /59 angles of attack and side slip begin to increase monotonically and if the rotation does not cease, then the aircraft may go into unallowably large angles and at higher dynamic heads even be destroyed. It is obvious that the smaller the value of the critical angular rolling velocity, the more probable will be its attainment in flight during rolling maneuvers. The determinant in this case is the least absolute value of the critical rolling velocity.
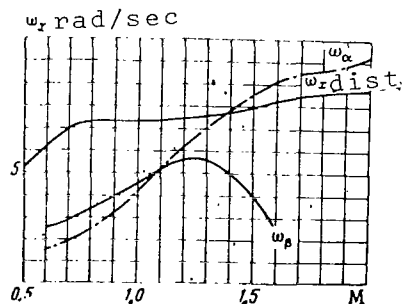
Fig. 2.11

Let us look at several characteristics of the aircraft motion in carrying out rolling maneuvers. As noted above as the angular rolling velocity approaches the critical value the degree of stability of motion of an aircraft is decreased. It is obvious in such case if certain aerodynamic moments act on the controls of the aircraft, it will then react more strongly than the excess stability, i.e., the nearer the angular

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rolling velocity will be to the critical. Furthermore, due to action of the inertial moments, the deflection of the rudder and the elevator will lead to simultaneous change in the angles of attack and side slip. In this case the motion of the aircraft, in a rolling turn, is quite similar to the motion of a gyroscope, which under the action of a disturbing moment begins to change its orientation, by precessing in a direction orthogonal to the action of the moment.

$w_x$crit
rad/sec

10

DH.
Vampire
Follend
"LET"

5

Douglas X-3
Lockheed F-104

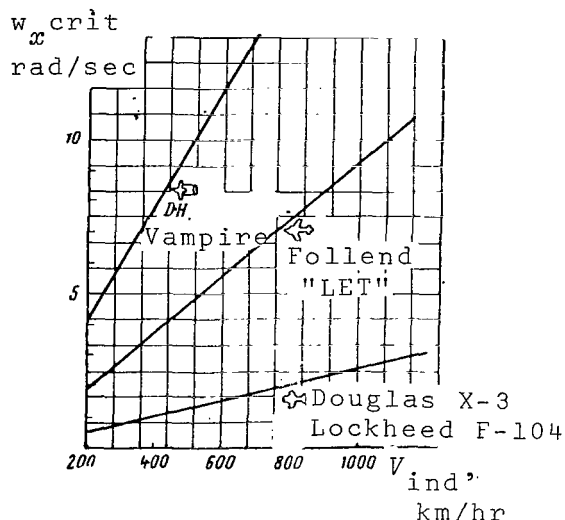200   400   600   800   1000 V_{ind},
km/hr

Fig. 2.12

Let us look in more detail at a simplified physical diagram of the motion of an aircraft in a rolling turn with an angular velocity less than the critical, with simultaneous deflection of the elevator. The deflection of the elevator leads to the appearance of an additional component of the angular velocity of rotation of the aircraft relative to the axis OZ_1, thanks to which the vector of the total angular velocity Ω ceases to be in agreement with the major inertial axis OX_1 (see Fig. 1.8). The deviation of the vector Ω from the axis OX_1 leads to the appearance of the moment from the centrifugal forces, equal to $(J_z - J_x)\omega_z\omega_x$. This moment begins to "turn" the aircraft into a yawing motion until it compensates the aerodynamic moment of the directional stability. In this case the deflection of the control stick and the creation of a positive angular pitching velocity ω_z, lead to the appearance of an angle of side slip of the same sign as the angular rolling velocity, i.e., with starboard rolling ($\omega_x > 0$) the right wing goes ahead ($\beta_x > 0$), and with port rolling ($\omega_x < 0$), the left wing goes ahead ($\beta < 0$). Rolling with recoil of the control stick after coming out of a negative G-force will lead to the development of side slip with a sign that is opposite to that of the angular rolling velocity, i.e., with starboard rolling ($\omega_x > 0$) the left wing goes ahead ($\beta < 0$) and vice versa.

In this stage of describing the dynamics of rolling maneuvers it follows to introduce a new, previously unconsidering factor, the presence of excess lateral stability m_x^β in the aircraft. The lateral stability of the aircraft leads to an angle of side slip, and in the general case the angle of attack also, beginning to substantially influence the angular rolling velocity, as a result of which the motion of the aircraft is significantly complicated.

Roughly speaking, we can define two characteristic types of rolling maneuvers in an aircraft. In the first case the effect of lateral stability leads to a decrease in the value of the angular rolling velocity, i.e., it is manifested in a seeming decrease in

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efficiency of the ailerons. The second case is characterized by directly opposite phenomena, a seeming increase in the efficiency of the ailerons all the way up to loss in the controllability of the aircraft by the ailerons, when the aircraft continues its rolling turn, regardless of the ailerons being in neutral position. Both types of rolling maneuvers may be accompanied by the development of large angles of attack and side slip and by large G-forces.

As an example of a rolling maneuver of the first type, let us look at a simplified physical picture of the motion of an aircraft with aileron deflection carried out under conditions of flying with a positive G-force. For simplicity we assume that the aircraft possesses a constant excess lateral stability, i.e., m_x^β = const. As noted above, in carrying out a rolling maneuver with the original positive G-force the angles of side slip develop with the same sign as that of the angular rolling velocity. A side slip of such sign creates a moment relative to the longitudinal axis OX_1, which impedes development of the angular rolling velocity due to the lateral stability m_x^β. In fact, from the equation of the equilibrium of moments relative to the longitudinal axis, the expression for the steady value of the angular rolling velocity may be written in the form

$$\overline{\omega}_x = -\frac{1}{\overline{m}_x^{\overline{\omega}_x}} \left[\overline{m}_x^\delta {}_a\delta_a + \overline{m}_x^\beta \cdot \beta \right]. \tag{2.45}$$

From expression (2.45) it follows that for the creation of a positive angular rolling velocity $\overline{\omega}_x > 0$, a positive moment from the ailerons ($\Delta\overline{m}_x = \overline{m}_x^\delta {}_a\delta_a > 0$) is required and, as noted above, the side slip developing in such case has a positive sign ($\beta > 0$) and consequently decreases the value of $\overline{\omega}_x$ (since $\overline{m}_x^\beta < 0$). From the view point of the pilot, a seeming decrease in the efficiency of the ailerons is observed. On the contrary, with deflection of the ailerons from the conditions of flight with a negative G-force side slip is developed with an angular rolling velocity that is more positive. Such a side slip facilitates the increase in rolling velocity and seems to increase the efficiency of the ailerons. Moreover, such a moment may occur when after deflection of the ailerons the developing side slip is found to be so large that without using the ailerons the aircraft maintains its rolling turn. In several instances resetting of the ailerons may in practice not lead to a change in the angular rolling velocity. This is an example of the second type of rolling maneuvers, which has been called by a number of different designations: autorotation, self-turning, inertial rotation, etc. Below we shall call it the system of inertial rotation.

CHAPTER III

STUDY OF THE MOTION OF AN AIRCRAFT USING THE METHODS OF THE QUALITATIVE THEORY OF DIFFERENTIAL EQUATIONS

11. Using the Methods of the Qualitative Theory of Differential Equations for Analyzing the Spatial Motion of an Aircraft

At the present time much scientific and technical literature has been published on the dynamics of aircraft, however practically all this literature has been devoted to a study of the motion of an aircraft with small disturbances, when linearization of the equations of motion is possible. In this respect the basic mathematical tool, on the basis of which in these papers the dynamics of an aircraft have been analyzed, is the theory of linear differential equations, the method of Laplace transforms, frequency methods, etc. To study the dynamics of aircraft in a general formulation, i.e., when large disturbances are taken into account and the non-linear equations of motion are analyzed, these methods become inapplicable. At the present time there are no analytical methods which will permit finding a solution to the nonlinear equations of motion of an aircraft. To describe the basic properties of solving these equations and determining their characteristics, in the present paper we shall use the methods of the qualitative theory of differential equations. However, we must note that the methods of the qualitative theory of differential equations are used mainly for analyzing second order equations and are significantly less developed for differential equations of a higher order. In the present paper an attempt is made to use several of the existing results, mainly for the purpose of classifying the possible types of spatial motions of an aircraft. Included in these results are, in first order, the general representations of the structure of the solutions to nonlinear differential equations, the concept of singular points, separatrix surfaces, etc. All the required information and formulations within the framework of the mathematical apparatus used in the paper are given in the present paragraph. To obtain more detailed information, the reader is referred to the special literature [1] - [8].

In the sections below we derive formulas for finding the parameters of motion of an aircraft at the singular points of the equations of motion, corresponding to the given deflection of the

controls (δ_e, δ_r, δ_a), and the aperiodic stability of such motions is determined. These formulas and conditions of stability of motion are used in practically all the remaining sections of the book in analyzing the controlled motions of an aircraft. On the basis of analyzing the dependence of the amount of deflection of the ailerons, required for carrying out the spatial maneuver, on the parameters of the longitudinal control, basic types of rolling maneuvers appear which differ in the reaction of the aircraft to the deflection of the ailerons, and the nature of the motion in the vicinity of the singular points is determined.

Let us assume as above that for the given time of motion of the aircraft, the speed and height of flight of the aircraft is practically constant and the influence of the forces of gravity can be neglected. Using such assumptions we can analyze Equations (1.28) the right hand parts of which are clearly independent of time. Such systems of equations belong to the so-called "autonomous systems" (sometimes called dynamic systems), for which the qualitative theory of differential equations has been mainly developed.

In general form the equations of motion of an aircraft can be written in the following manner:

$$\dot{x}_i = X_i(x_1 \ldots x_n);$$
$$i = 1, \ldots n. \tag{3.1}$$

The variables ($x_1, \ldots x_n$) are analyzed as coordinates of the point of n-dimensional phase space and may represent the values of the angular velocities, the angles of attack and side slip of the aircraft, etc. The methods of the qualitative theory of differential equations can be simply represented in the following manner. Since in the right hand parts of Equations (3.1) the time does not appear in explicit form, then it can always be excluded from the system of differential equations and in the same manner they can be reduced by an order under the condition that not all the right hand parts of the equations vanish. As a result of such an operation, differential equations are obtained which do not contain time, the integrals of which determine the relationship between the phase coordinates ($x_1, \ldots x_n$), that are satisfied throughout the time of the motion. The pictures of the motion obtained in such case on the phase plane for the equations of second order, or in phase space if the order of the equations is higher than the second, clearly describe the motion; in particular they indicate the regions of stable and unstable motion, periodic motions, etc. The practical fulfillment of plotting the trajectory of the motion in phase space /64 described above in the majority of cases is more complicated than finding solutions to the original equations in general form. However, for equations of second order, approximate methods do exist for plotting such trajectories, and have been described in detail in the literature [1] - [3]. We shall not pause to analyze these, since their use for equations higher than second order is either

quite complex or in general impossible. Additional problems in plotting phase trajectories for equations of a higher order are associated with the practical impossibility of a graphic representation of space with order higher than the third, in connection with which a slight loss appears in the clarity of the results.
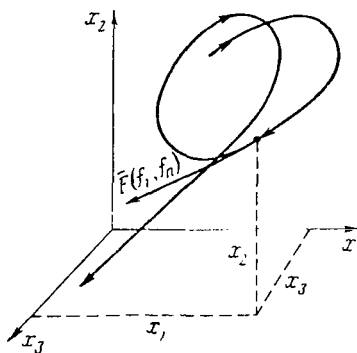

Fig. 3.1.

The basic representations of the character of the phase trajectories can be obtained if we look at the right hand parts of the system of Equations (3.1), as the components of the vector $\overline{F}(f, \ldots f_n)$, which leave the point $(x_1, \ldots x_n)$ in Figure 3.1. Thus, the system of Equations (3.1) can be set in correspondence with a certain vector field. In this vector field the vector $\overline{F}$ determines the phase velocity, since its components are the values of the derivatives of the coordinates. At all points of space the phase trajectories are directed tangent to the vector $\overline{F}$. On this are based several approximate methods of plotting the phase trajectories. In fact, starting at a certain point of phase space $(x_1', \ldots x_n')$, using Equations (3.1) we can compute the components of the vector $\overline{F}$ and it is shifted a slight distance in the direction of the vector. Sequentially repeating this procedure in sufficiently small "steps" of change in the parameters $(x_1, \ldots x_n)$, we can plot the phase trajectory with the necessary accuracy.

Both for problems of stability and for the overall qualitative analysis of differential equations, considerable attention is given to the study of singular points or points of rest of the system of Equations (3.1). These are the points $(x_1^{(0)}, \ldots x_n^{(0)})$, for which all right hand parts of the equations vanish, i.e.,

$$X_i(x_1^{(0)}, \ \ldots \ x_n^{(0)}) = 0;$$
$$i = 1, \ \ldots \ n. \tag{3.2}$$

The type of motion in the vicinity of the singular points can be obtained by finding the solution to the linearized equations relative to the parameters of motion at this point. The motion in the vicinity of the singular points for the system of equations of second order have been studied in detail so that it is impossible to speak of equations of a higher order. Before we proceed to a description of the properties of the solutions in the vicinity of the singular points in the n-dimensional case, let us narrow down the class of systems to be analyzed, i.e., let us assume that the mechanical problems which we are studying belong to the so-called "rough systems". The concept of rough systems [11] includes the following. In order for the analyzed dynamic model of motion to be

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well reflected in the properties of the real physical process, it is necessary that it be immune to small changes in the parameters. First of all, in dynamic systems which represent the physical problems, with small changes in the parameters of the right hand parts of the equations, the qualitative structure of the separation into trajectories in phase space must remain constant. The systems, for which this requirement is satisfied, are called rough systems [11]. It is obvious that in rough systems all groups of roots of the characteristic equation are possible with the exception of the purely imaginary and idential (multiple) roots.

In fact, when the roots are purely imaginary, which corresponds to an undamped oscillational motion, in the general case of a sufficiently small change in the parameters, in order for the real part to appear in the roots, the oscillations become either damped or divergent.

In determining the dependence of the type of singular points on the parameters of the system of equations of motion, singular points can be encountered which do not satisfy the condition of a rough system. The motion of an aircraft in the vicinity of the nonrough singular points of practical interest is not presented since the probability of such a combination of parameters of an aircraft is in fact zero.

The book by Andronov, Vitt and Khaikin [11] - cites the classification of possible types of singular points in rough systems for equations of second order, the bases of which are the saddle point and the stable and unstable focal points. By expanding these concepts to the equations of n-th order, we can classify the singular points according to the type of curve of the characteristic equation obtained in linearizing the original equations of motion relative to the values of the parameters of motion at the singular point.

Let us introduce, in analogy with Reference [11], the following definitions of the singular points as a function of the roots of the characteristic equation of the linearized system of equations.

1. All Real Roots of the Characteristic Equation are Negative.

(a) Stable spatial focus. The singular point will be a stable spatial focus if there are complex-conjugate roots with a negative real part.

(b) Unstable spatial focus. The singular point will be an unstable spatial focus if there is even one pair of complex-conjugate roots with a positive real part.

(c) Stable spatial point. The singular point will be a stable /66 point if all roots are real and negative.

2.   Even One Positive Real Root of the Characteristic Equation Exists.

In this case the singular points, regardless of the type of other roots, will be termed a spatial saddle point.
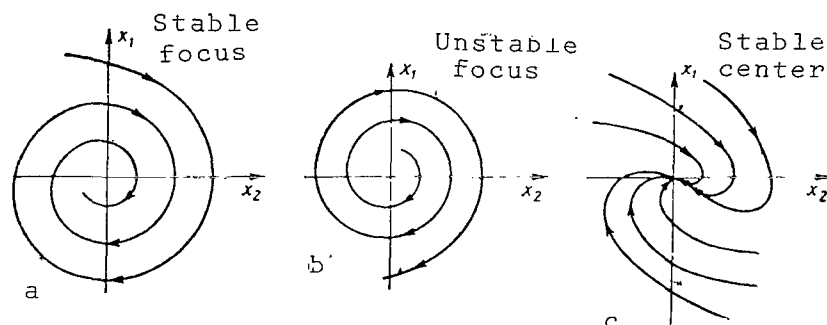
Fig. 3.2

In cases (a) and (b) the phase curves come as near as desired to the origin when $t \to \infty$ (stable singular points). The motion in the vicinity of the singular points in cases (a),(b) and (c) has the character of damped or divergent oscillations. Figure 3.2 shows examples of phase trajectories for the equation of second order in these cases. The arrows indicate the direction of motion of the figurative point for the phase trajectory with increase in time.

A characteristic of the motion in the vicinity of the singular point of the "spatial saddle" type is the fact that almost all phase trajectories approach at a certain minimal distance to the singular point when $0 \leq t \leq \infty$ and does not reach it, i.e., all integral curves are saddle points. The exception involves phase trajectories which lie on certain singular surfaces which may either "enter" the singular point or "leave" it. The diagram of the phase motion in the vicinity of the saddle singular point for the equation of second order is shown on Figure 3.3.

Let us look at this latter case in greater detail. As noted above, the solution for any parameter of motion in the vicinity of the singular point for a rough system [11] can be found from the system of uniform linear differential equations and are written in the form

$$x_i = A_{1i}e^{\lambda_1 t} + A_{2i}e^{\lambda_2 t} + \ldots$$
$$i = 1, \ldots n. \tag{3.3}$$

Let us determine the method of finding the equations of the

separatrix surface. The basic property of the separatrix surface is that the solutions for the phase coordinates which lie on this surface are independent of all the real roots of the characteristic equation. Because of this, stable solutions may exist for the separatrix surface in particular although in all the remaining space they are aperiodically unstable. This will be so in the case when the solution for the phase trajectories on the separatrix surface does not depend on the positive real root, leading in the entire remaining phase space to aperiodic instability of motion.

"Saddle"

Fig. 3.3

Let λ_1 be a real root not having a multiple. In order that the solution for all the parameters of motion x_i be independent of this root it is necessary and sufficient that all coefficients A_{1i} when exp in the power $\lambda_1 t$ is equal to zero. Let us show that for all the coefficients A_{1i} to vanish, it is usually sufficient for the coefficient A_{1i} to vanish in the solution of even one of the variables (see also [18]). In fact let us look at a system of linear equations that describe the motion of an aircraft in the vicinity of the singular point:

$$
\left.
\begin{aligned}
\dot{x}_1 &= a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n; \\
\dot{x}_2 &= a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n; \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
\dot{x}_n &= a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n.
\end{aligned}
\right\}
$$

(3.4)

Let there be known the real root λ_1 which is the solution to the characteristic equation. If the solution $x_1 = A_1^0 \exp(\lambda_1 t)$ is placed into the system of Equations (3.4), the characteristic determinant then vanishes, thus indicating that one of the equations of System (3.4) is a consequence of the other equations. If we exclude one of the equations, for example, the first of the system /68 of Equations (3.4) we find a system of linear algebraic equations for determining the components of the solutions $x_2, \ldots x_n$ for the term exp $\lambda_1 t$ in the decision function x_1. Since the root λ_1 does not have a multiple then we can always exclude one equation from the System (3.4) in order for the remaining equations to have the characteristic determinant not equal to zero, and consequently because of the linearity of the equations, there would exist the equation

$$
x_j = A_1^0 \cdot g_j(a_{jk}, \lambda_1) e^{\lambda_1 t},
$$

(3.5)

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i.e., the solution would be represented in the form of the product of the coefficient A_1^0 times a constant which depends on the number of the parameter of motion j, in which it is computed also from the parameters of the characteristic equation.

The value of A_1^0 is a function of the initial conditions. Since the equations are linear and uniform, then A_1^0 is written in the form

$$A_1^0 = \sum_{i=1}^{n} \cdot b_i x_i(0).$$ (3.6)

As follows from Expression (3.5) in that case when $A_1^0 = 0$, all coefficients vanish with the respective terms exp $(\lambda_1 t)$. Therefore, if in the solution for any variable the coefficient for the term exp $(\lambda_1 t)$ vanishes, then the respective coefficients for this term in the solutions for the other variables will also vanish. The exception to this will be the degenerate cases when the coefficient for exp $(\lambda_1 t)$ is identically equal to zero in the solutions for several variables, for example, if the respective value $g_i \equiv 0$.

From expression (3.6) it follows that the region of the initial values, for which the term exp $(\lambda_{\varkappa} t)$ is absent in the solution, is the n-dimensional plane, which passes through the singular point. The equation of this plane can be found if the expression for A_1^0 is equated to zero [see Expression (3.6)]:

$$\sum_{i=1}^{n} b_i x_i = 0.$$ (3.7)

Because of the uniqueness of the solution to the system of differential Equations (3.4) all integral curves, having even one point other than the singular point in common with the points of the plane (3.7), lie in it completely. Hence it follows that none of the integral curves intersect this plane, i.e., the plane is a separatrix surface, which separates the phase space into regions. The number of separatrix surfaces, which pass through the singular point, is equal to the number of real roots. If the characteristic equation of the system has only one positive real root, then the integral curves in the entire phase space, with the exception of the integral curves lying on the separatrix surface, approach a certain minimal distance to the singular point, at which position they remain (see Fig. 3.3). An exception is the integral curves which lie on the separatrix surface and approach as close as may be desired to the singular point if the real parts of all the remaining roots are negative. If the characteristic equation has several separatrix planes, then each intersection of them is an $(n - 1)$-dimensional surface (plane) of first order.

With distance from the singular point, the accuracy of representing the equations of motion as linear deteriorates and the separatrix n-dimensional planes change into surfaces of a more complex type, separating the space into several sub-spaces, in each of which may appear different types of integral curves.

The basic elements which determine the qualitative picture of the distribution of integral curves in phase space for rough dynamic systems, are the singular points and the separatrix surfaces. If the position and types of the singular points and separatrix surfaces are known (as noted above, the separatrix surfaces may intersect at the singular points of center and saddle type) then the qualitative picture of the distribution of integral curves in the phase space can be shown in general outline and the picture of the motion can be determined as a function of the initial conditions.

Change in the parameters of the equations of motion (and in studying the dynamics of an aircraft for constant conditions of flight the values of the control deflections δ_e, δ_a, δ_r) will lead to a change in the integral curves. In such case the overall form of the integral curves may undergo only slight quantitative changes, i.e., the topological structure of separating the phase space (the number and character of the singular points and the separatrix surfaces) may not change. Only for certain singular so-called "bifurcation" values of the parameters do there exist qualitative changes in the phase picture of the distribution of integral curves. This may be expressed in the change in the type of singular points (for example, conversion of a stable-focus type singular point into a saddle point), in the change in the number of singular points, etc. It is obvious that such values of the parameters of control are of special interest, since they determine the boundaries at which a qualitative change in the process of aircraft motion occurs, for example, regions of unstable motion, etc., appear.

In further studies we shall determine the dependence of the type of singular points on the parameters of aircraft motion, in first order, on the values of the control deflections δ_e and δ_r, in particular, we shall seek the bifurcation values of these parameters.

12. Determining the Parameters of Controlled Motion of an Aircraft at the Singular Points. Formulas for the Static Solutions.

Let us proceed to an analysis of the qualitative character- /70
istics of spatial motion of an aircraft by taking into account the limiting assumptions introduced above. The basic characteristics in the dynamics of an aircraft which can not be determined from the simplified linearized equations, appear in those cases when the motion of an aircraft is accompanied by strong rolling. It should be noted that the necessity for analyzing the total equations of motion arises not only in investigating rolling maneuvers which are accomplished by deflection of the ailerons. With the large values of excess lateral stability that are characteristic of modern aircraft, significant angular rolling velocities may develop as a

reaction to the deflection of the rudder and even when the aircraft enters into a wind blast.

The purpose of the investigations includes finding the relationship between the values of the angles of control deflection of an aircraft (δ_a, δ_e, δ_r) and those changes in the parameters of its motion (the angles of attack and side slip α, β and the projections of the vector of angular velocity ω_x, ω_y, ω_z) to which these deflections lead.

Isolated longitudinal and lateral motion of an aircraft is characterized by the fact that with deflection of the elevator and rudder at certain constant angles (δ_{e0}, δ_{r0}) the aircraft changes respectively the angle of attack and side slip to a certain constant value and begins to turn in flight relative to inertial space at a given angular velocity ($\bar{\omega}_z$, $\bar{\omega}_y$). In those cases when the maneuver involves a short segment of time or is carried out with large G-forces, the influence of gravity on the motion relative to the center of mass can be ignored and the limiting values of the angular velocities of the aircraft may be assumed to be constant values. Such steady limiting conditions of flight correspond to the singular points of the equations of motion of an aircraft with the gravitational terms dropped. Because of such a relationship between the deflection of the controls and the parameters of motion, investigation of the dynamics of an aircraft when the maneuver is carried out by deflection of the controls in various combinations at certain constant angles, can be accomplished most completely and clearly by using the methods and terminology of the qualitative theory of differential equations.

When the problem of the relationship between the deflections of the controls and the changes in the parameters of motion was solved on the assumption of smallness of such changes, and the dynamics of the aircraft were described by a system of linear differential equations such as (1.34) and (1.35), the solution was rather simple. The analytical methods of such motions are discussed in detail in a number of papers on the dynamics of an aircraft ([18], [19], etc.). The results of these investigations (for those cases when the influence of the gravitational terms can be ignored) are reduced to three basic points: /71

1. The reaction of an aircraft to deflection of the elevator (δ_e) does not depend on deflections of the ailerons and rudder. Analogously, the reaction of an aircraft to deflection of the ailerons and the rudder does not depend on deflections of the elevator. In other words, the longitudinal and lateral controlled motions of an aircraft are independent when the changes in the parameters of motion are small.

2. The quantities of the steady values of the angles of attack and side slip, and also the projections onto the body axes of the vector of angular velocity ($\bar{\omega}_x$, $\bar{\omega}_y$, $\bar{\omega}_z$) are unique functions of the

values of deflections of the controls, i.e., the steady values of all the parameters of motion are independent of the initial conditions and the sequence of deflections of the controls and are only possible with a given deflection of the controls.

3.   The dependence of the parameters of the disturbed motion of an aircraft on the values of deflections of the controls in the case of the linear aerodynamic characteristics is a linear one, for example, a change by two times of the angle of deflection of the rudder relative to the balanced position, corresponding to horizontal flight, leads to a change in the value of the angular velocity ω_z also by two times.

In analyzing the spatial motions accompanied by strong rolling, strictly speaking, none of these conclusions is retained. In first order this involves maneuvers with which simultaneous strong deflections of the ailerons and rudder are carried out. Below it will be shown that changes in all the parameters of motion of an aircraft with such maneuvers are interrelated. Moreover the values of the limiting steady values of the angles of attack and side slip and the projections of the vector of angular velocity (ω_x, ω_y, ω_z) are not unique functions of deflections of the controls. Mathematically this means that for each combination of deflections of the controls there are several singular points of the system of equations of motion. In all these cases the linear character of the dependence of angular rolling velocity on the value of deflection of the ailerons is disrupted.

We must note that the conclusions made above were obtained from linearized equations and in the engineering sense are valid for small deflections of the controls and naturally follow from the total equations of motion. In fact, regardless of the fact that the system of equations of motion of an aircraft has several singular points for each combination of the control deflections, small deflections in the controls and small external disturbances can not "carry off" the parameters of motion of an aircraft sufficiently far from the region of "attraction" of the basic singular point, which is determined quite precisely by using simplified linearized equations.

In the problems of studying the motions described by nonlinear equations, we isolate several basic questions, which should be examined sequentially to simplify the problem.

1.   Finding all possible combinations of steady values of the parameters of motion, i.e., finding the entire complex of singular points corresponding to given values of the disturbances, deflections of controls, etc.

2.   Investigating the type of motion in the vicinity of each singular point and its stability (motion "in the small").

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3. Investigating the motion in the entire phase space (motion "in the large").

In the present paragraph let us look at the first part of the overall problem and determine the dependence of the values of the parameters of motion at the singular points on the values of deflections of the controls, i.e., the "trajectory of the singular points". Knowledge of these functions permits, for any combination of deflections of the controls, determining all singular points, i.e., all possible values of the "points of rest" - the "static solutions". It should be noted that there exists a theoretical difference between the steady motion of an aircraft and the parameters of motion representing the singular points. This distinction consists in the singular points describing all possible "points of rest" of the system of differential equations of motion, whereas the steady systems of motion of an aircraft exist only for stable singular points toward which the phase trajectory approaches in the limit.

Let us proceed to finding the values of the parameters of motion at the singular points. For this purpose let us first rewrite the system of Equations (1.28) in a form which is convenient for analysis:

$$
\begin{aligned}
\alpha' &= \mu \bar{\omega}_z - \frac{c_y^\alpha}{2} \alpha - \mu \bar{\omega}_x - \frac{c_y^{\delta e}}{2} \delta_e \cdot \\
\bar{\omega}_z' &= \bar{m}_{zb}^\alpha \cdot \alpha + \bar{m}_{zb}^{\omega z} \cdot \bar{\omega}_z - A \mu \bar{\omega}_x \bar{\omega}_y + \bar{m}_{zb}^{\delta e} \cdot \delta_e \cdot \\
\beta' &= \mu \bar{\omega}_y + \frac{c_z^\beta}{2} \beta + \mu (\alpha + \alpha_0) \bar{\omega}_x + \frac{c_z^{\delta r}}{2} \cdot \delta_r \\
\bar{\omega}_y' &= \bar{m}_y^\beta \cdot \beta + \bar{m}_y^{\omega y} \cdot \bar{\omega}_y + \mu B \bar{\omega}_x \cdot \bar{\omega}_z + \bar{m}_y^{\delta r} \delta_r \\
\bar{\omega}_x' &= \bar{m}_x^\beta \cdot \beta + \bar{m}_x^{\omega x} \cdot \bar{\omega}_x - \mu C \bar{\omega}_y \bar{\omega}_z + \bar{m}_x^{\delta a} \delta_a
\end{aligned}
\right\}
\qquad (3.8)
$$

To obtain values of the variables at the singular points in correspondence with the definition (see Section 11), let us equate the right hand parts of Equations (3.8) to zero. Equating the right hand parts of Equations (3.8) to zero indicates that all derivatives of the parameters of motion are equated to zero and consequently corresponds to the extreme motion or "static" solution, which generally speaking may also not be stable. To find the "coordinates" of the singular points in phase space we must find /73 all solutions for the parameters of motion which satisfy the nonlinear system of algebraic equations obtained. In those cases when the aerodynamic coefficients are linear functions of their own arguments, the system of algebraic equations may be easily solved in parametric form, if we take the value of the angular rolling

velocity $\bar{\omega}_x$ for the parameter, since all nonlinear terms of the equations are functions only of this parameter. In the general case of nonlinear dependences of the aerodynamic coefficients on the parameters of aircraft motion, the solution to the obtained system of nonlinear algebraic equations is additionally complicated. In this case it can be solved by various iteration methods.

The system of algebraic equations obtained by equating the derivatives to zero, in the general case is not uniform and has the following form:

$$
\left.
\begin{aligned}
&\frac{c_y^\alpha}{2}\,\alpha_{s\,s} - \mu\bar{\omega}_z{}_{s\,s} + \mu\bar{\omega}_x \cdot \dot{\beta}_{s\,s} = -\frac{c_y^{i_e}}{2}\,\delta_e; \\[4pt]
&\overline{m}_{z\,b}^\alpha \alpha_{s\,s} + \overline{m}_{z\,b}^{\omega_z}\bar{\omega}_{z\,s\,s} - \mu\,A\,\bar{\omega}_x\bar{\omega}_{y\,s\,s} = -\overline{m}_{z\,b}^{i_e}\cdot\delta_e; \\[4pt]
&\mu\bar{\omega}_x\cdot\alpha_{s\,s} + \frac{c_z^\beta}{2}\,\beta_{s\,s} + \mu\bar{\omega}_{y\,s\,s} = -\frac{c_z^{i_r}}{2}\,\delta_r - \mu\bar{\omega}_x\cdot\alpha_0; \\[4pt]
&B\mu\bar{\omega}_x\cdot\bar{\omega}_{z\,s\,s} + \overline{m}_y^\beta\cdot\beta_{s\,s} + \overline{m}_y^{\omega_y}\cdot\bar{\omega}_{y\,s\,s} = -\overline{m}_y^{i_r}\cdot\delta_r.
\end{aligned}
\right\} \qquad (3.9)
$$

If we assume $\bar{\omega}_x$ to be a parameter whose values can be given arbitrarily, then from expressions (3.9) it is clear that in the right hand parts of the system of algebraic equations we find terms which depend not only on the values of the control deflections (δ_a, δ_e, δ_r) relative to their original position, corresponding to trim in steady horizontal flight without rolling, but also on the angle between the major inertial axis and the vector of flying speed of the aircraft (α_0) at the beginning of the maneuver and on the size of the angular rolling velocity $\bar{\omega}_x$. With a fixed value of $\bar{\omega}_x$ the system of equations becomes linear and each combination of parameters in the right hand sides of the system of Equations (3.9) is represented by a unique complex of parameters of motion of the aircraft at the singular point. With a continuous change in the value of any of the parameters of control, in the right hand parts of the equations we find the "trajectory" of the coordinates of the singular points as a function of the respective parameter.

From the system of algebraic Equations (3.9) it follows that the parameters of motion at the singular points depend linearly on the size of the deflections of the elevator and rudder (δ_e, δ_r), however the coefficient of proportionality is determined by the size of the angular rolling velocity $\bar{\omega}_x$. Such a linear dependence permits representing the values of the parameters of motion at the singular points which will be denoted by the subscript "st" (static solution), in the following form:

$$\alpha_{s\,s} = A^{\alpha}_{c^{\delta_e}_y} \cdot c^{\delta_e}_y \cdot \delta_e + A^{\alpha}_{m\delta_e} \cdot \Delta\overline{m}_z + A^{\alpha}_{c^{\delta_r}_z} c^{\delta_r}_z \cdot \delta_r + \\
+ A^{\alpha}_{\alpha_o} \cdot \alpha_0 + A^{\alpha}_{my\delta_r} \cdot \Delta\overline{m}_y;$$

$$\overline{\omega}_{z\,s\,s} = A^{\overline{\omega}_z}_{c^{\delta_e}_y} \cdot c^{\delta_e}_y \cdot \delta_e + \ldots;$$

$$\beta_{s\,s} = A^{\beta}_{c^{\delta_e}_y} \cdot c^{\delta_e}_y \cdot \delta_e + \ldots;$$

$$\overline{\omega}_{y\,s\,s} = A^{\overline{\omega}_y}_{c^{\delta_e}_y} \cdot c^{\delta_e}_y \cdot \delta_e + \ldots;$$

$$(3.10)$$

(All formulas for the values of A^{α}, $A^{\overline{\omega}_z}$, A^{β}, A^{ω_y} are tabulated in Table 2). The formulas shown in Table 2 were obtained by solving the non-uniform system of algebraic Equations (3.9). The solution for the i-th variable, as we know from linear algebra, is found from the formula

$$x_i = \frac{|\Delta_{if}|}{\Delta_0},$$

$$(3.11)$$

where $|\Delta_0|$ is the characteristic determinant of the system of equations, and $|\Delta_{if}|$ is the determinant composed of those elements in which the i-th column is substituted by the column of right hand parts.

From the method itself of obtaining Equations (3.9), when the derivatives are equated to zero, it follows that the equation

$$|\Delta_0| = A_0,$$

$$(3.12)$$

is satisfied, where the coefficient A_0 is the free term of the characteristic Equation (2.10), obtained in analyzing the stability of the steady rolling turn of an aircraft ($\overline{\omega}_x$ = const).

The first thing to be noted in Equation (3.10) is that of the property of superposition of the solutions obtained for the various disturbances included in the right hand sides of Equations (3.9). This property is due to the algebraic Equations (3.9) being linear in the analyzed formulation for the linear aerodynamic coefficients when $\overline{\omega}_x$ = const, and consequently any solution can be represented in the form of a sum of the solutions as a function of each individual disturbance. In the general case of nonlinear aerodynamic coefficients [for example, for a dependence of the type $m_z^{\alpha}(\alpha)$, $m_y^{\beta}(\alpha)$, /75
etc.], this property will not be satisfied. Let us note however that all nonlinear dependences of the aerodynamic coefficients included in the equation of moments relative to the longitudinal axis (OX_1) obviously do not influence formulas (3.10) since only the

TABLE 2

q_{SS}	$\bar{\omega}_{YSS}$	
$\dfrac{1}{2A_0}\left(-\bar{m}_z^{\bar{\omega}z}\,\dfrac{c_z^{\beta}}{2}\,\bar{m}_y^{\bar{\omega}y}-\mu\bar{\omega}_x^2\,AB\,\dfrac{c_z^{\beta}}{2}+\mu\bar{m}_y^{\beta}\,\bar{m}_{z\flat}^{\bar{\omega}z}\right)$	$-\dfrac{1}{2A_0}\dfrac{c_z^{\beta}}{2}\left(\mu B\bar{\omega}_x+\mu\bar{\omega}_x\bar{m}_{z\flat}^{\bar{\omega}z}\,\bar{m}_y^{\beta}\right)$	$c_y^{\delta}e.\,r_{\ni}$
$\dfrac{-1}{A_0}\left(-\mu\dfrac{c_z^{\beta}}{2}\,\bar{m}_y^{\bar{\omega}y}+\mu\bar{3}\bar{\omega}_x^2\,B+\mu^2\bar{m}_y^{\beta}\right)$	$\dfrac{1}{A_0}\bar{\omega}_x^3+\mu^2\,\bar{\omega}_x m^{\beta}-\mu B\bar{\omega}_x\,\dfrac{c_z^{\beta}}{2}\,\dfrac{c_y^{\alpha}}{2}$	$\bar{m}_z^{\delta}e.\,r_e$
$\dfrac{\mu\bar{\omega}_x}{A_0}\left(-\mu^3\bar{\omega}_x^3\,AB-\mu\bar{\omega}_x\bar{m}_y^{\bar{\omega}y}\,\bar{m}_{z\flat}^{\bar{\omega}z}-\mu^2 A\bar{\omega}_x\bar{m}_y^{\beta}\right)$	$\dfrac{\mu\omega_x}{A_0}\bar{m}_z^{\bar{\omega}z}\,\bar{m}_y^{\beta}+\mu^2\bar{\omega}_x^2 B\bar{m}_z^{\alpha}j_{\flat}+\mu\bar{m}_y^{\beta}\cdot\bar{m}_{z\flat}^{\alpha}$	$a_0\, t\,o\,t$
$\dfrac{1}{2A_0}\left(-\mu^3\bar{\omega}_x^3\,AB-\mu\bar{\omega}_x\bar{m}_y^{\bar{\omega}y}\bar{m}_z^{\bar{\omega}z}-\mu^2 A\bar{\omega}_x\bar{m}_x^{\beta}\right)$	$-\dfrac{1}{2A_0}\bar{m}_{z\flat}^{\bar{\omega}z}\,\bar{m}_y^{\beta}+\mu\bar{\omega}_x^2 b\,\bar{m}_{z\flat}^{\alpha}+\mu\bar{m}_y^{\beta}\,\bar{m}_{z\flat}^{\alpha}$	$c_z^{\delta}r.\,r_r$
$\dfrac{1}{A_0}\left(\mu^2\bar{\omega}_x\bar{m}_{z\flat}^{\bar{\omega}z}+\mu^2 A\,\bar{\omega}_x\,\dfrac{c_z^{\beta}}{2}\right)$	$-\dfrac{1}{A_0}\bar{m}_{z\flat}^{\bar{\omega}z}\,\dfrac{c_z^{\beta}}{2}-\mu^2\bar{\omega}_x^2\,\bar{m}_{z\flat}^{\bar{\omega}z}+\mu\bar{m}_{z\flat}^{\alpha}\cdot\dfrac{c_z^{\beta}}{2}$	$\bar{m}_y^{\delta}r.\,r_r$
$\dfrac{\bar{\omega}_x}{A_0}\left(\mu^2\bar{\omega}_x\bar{m}_{z\flat}^{\bar{\omega}z}+\mu^2 A\bar{\omega}_x\,\dfrac{c_z^{\beta}}{2}\right)$	$-\dfrac{\bar{\omega}_x}{A_0}\bar{m}_{z\flat}^{\bar{\omega}z}\,\dfrac{c_z^{\beta}}{2}-\mu^2\bar{\omega}_x^2\,\bar{m}_z^{\bar{\omega}z}+\mu\bar{m}_z^{\alpha}\,\dfrac{c_z^{\beta}}{2}$	$\bar{m}_y^{\bar{\omega}x}$

$$A_0=\mu^2\left\{\left[\left(-\bar{m}_{z\flat}^{\alpha}-\dfrac{c_y^{\gamma}}{2\mu}\,\bar{m}_z^{\bar{\omega}z}\right)-\right.\right.$$

$$\mu=\dfrac{2m}{\varrho_{sl}}\,;\qquad A=\dfrac{J_y-J_x}{J_z}$$

$$\bar{\omega}_x=\omega_x\dfrac{l}{2V}\,;\qquad B=\dfrac{J_z-J_x}{J_y}$$

value $\bar{\omega}_x$ is included in it and the parameters from the equation of moments relative to the axis OX_1 are not.

Each of the coefficients, A^α, $A^{\bar{\omega}z}$, etc., represent the ratio of a certain polynomial in powers of $\bar{\omega}_x$ to the free term of the characteristic equation A_0. The coefficient A_0 (see Section 8) can vanish at the boundaries of the regions of stability of aircraft motion during rotation when $\bar{\omega}_x$ = const and there are usually two values of the angular rolling velocity. From the formulas in Table 2 it follows that with these values of the angular rolling velocity all the functions A^α, $A^{\omega z}$, A^β, $A^{\omega y}$ grow without limit, and with passage through the critical angular rolling velocity undergo disruption and change sign.

The physical sense of the increase in the coefficients of proportionality, A^α, A^β, etc., with approach to the critical velocity may be explained using the following simple arguments. A lowering in the value $A_0(\omega_x)$ means a decrease in the degree of stability of motion of the aircraft. Since the value of the disturbance is retained (a constant control deflection), then the reaction of the aircraft to this disturbance with lowering of the degree of stability increases and with critical rolling velocities, when the aircraft no longer possesses stability, this disturbance leads to an unlimited growth in all the parameters of motion of the aircraft. The character of the dependence of the basic terms of the functions A^α, $A^{\omega z}$, A^β, $A^{\omega y}$ on the angular rolling velocity is illustrated on Figures 3.4, 3.5 and 3.6.

For a full determination of the dependence of the parameters of motion of an aircraft at the singular points on the deflections of all controls (δ_a, δ_e, δ_r) we must exclude the value of the angular rolling velocity by using the respective dependence of $\bar{\omega}_x$ on the amount of control deflection. However such a method involves awkward computations, leads to poor results and is thus not rational. It is much more convenient to retain the dependence on $\bar{\omega}_x$ of the parameters of motion of an aircraft after determining them by an equation which permits finding the value of the required aileron deflection, corresponding to the motion of an aircraft with given values of $\bar{\omega}_x$, α_{ss}, $\bar{\omega}_{zss}$, β_{ss} and $\bar{\omega}_{yss}$. The value of the aileron deflection required for guaranteeing motion of an aircraft with an angular rolling velocity $\bar{\omega}_x$ is found from the equation of equilibrium of moments relative to the longitudinal axis in steady motion, which can be written in the following approximate form:

$$\Delta\bar{m}_x = \bar{m}_x^{\delta a}\cdot\delta_a = -\bar{m}_x^{\beta}\beta_{ss} - \bar{m}_x^{\bar{\omega}_x}\cdot\bar{\omega}_x + C\bar{\omega}_{yss}\bar{\omega}_{zss}. \qquad (3.13)$$

Since all the parameters of motion included in Equation (3.13) /76 are known functions of the angular rolling velocity $\bar{\omega}_x$ and of the values of the control deflections, then the value $\Delta\bar{m}_x$ can be easily found.
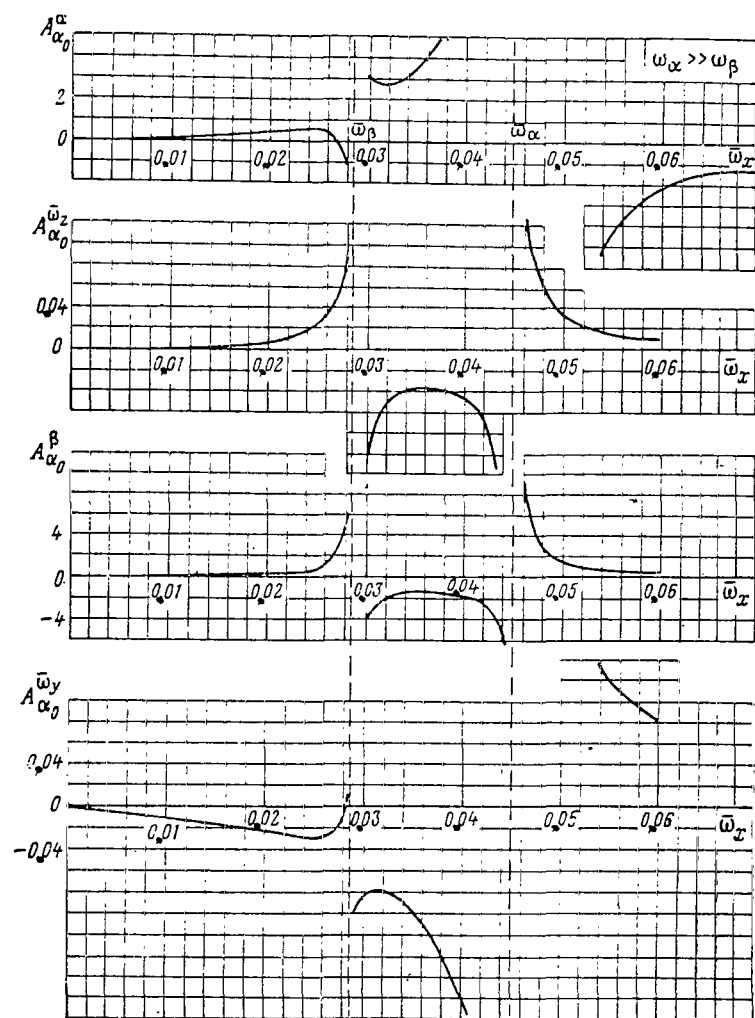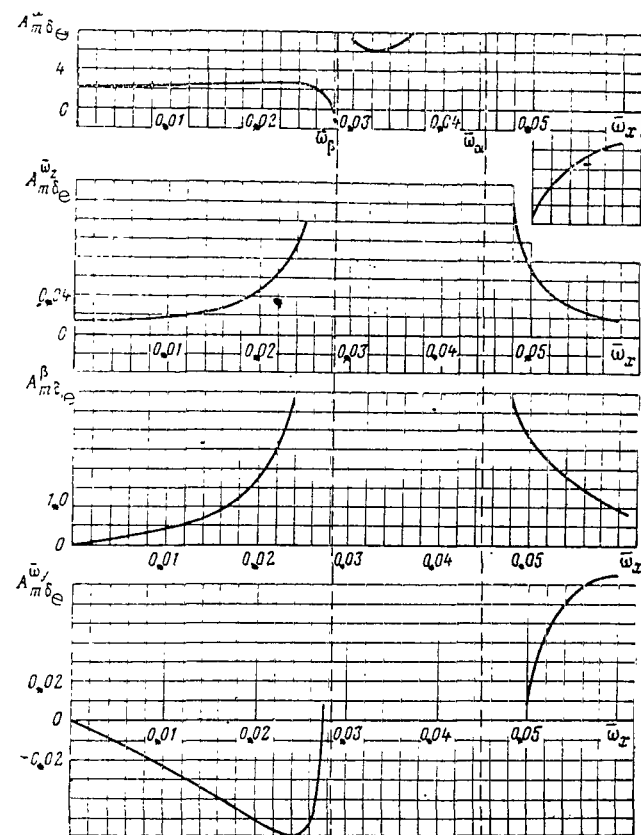
Fig. 3.4.

Fig. 3.5.

Figures 3.7 and 3.8 show examples of the dependences of the steady values of the basic parameters of motion of an aircraft on the angular rolling velocity $\bar{\omega}_x$. Graphs of static solutions for a rolling maneuver, carried out from horizontal flight, are plotted on Figure 3.7 and those for a rolling maneuver carried out from the conditions of flight with negative G-force, $\Delta \bar{m}_y = \Delta \bar{m}_z = 0$, are plotted on Figure 3.8.

It should be noted that the curves of the static solutions are plotted only for positive values of the angular rolling velocity ($\bar{\omega}_x > 0$). From the formulas in Table 2 it follows that all the functions A^β, $A^{\omega y}$ are odd and the functions A^α and $A^{\omega z}$ are even for $\bar{\omega}_x$, i.e., the curves for $\bar{\omega}_x < 0$ in the first case are antisymmetric to the curves for $\bar{\omega}_x > 0$ and may be obtained as an inverted mirror reflection of the curves for $\bar{\omega}_x > 0$, and in the second case - as a mirror reflection.

It is clear from Figures 3.7 and 3.8 that the obtained dependences on $\bar{\omega}_x$ of the parameters of motion of an aircraft have singularities when "critical" values of the angular rolling velocities exist. At these points all static solutions take infinitely larger values, whence it follows that the required aileron deflections for
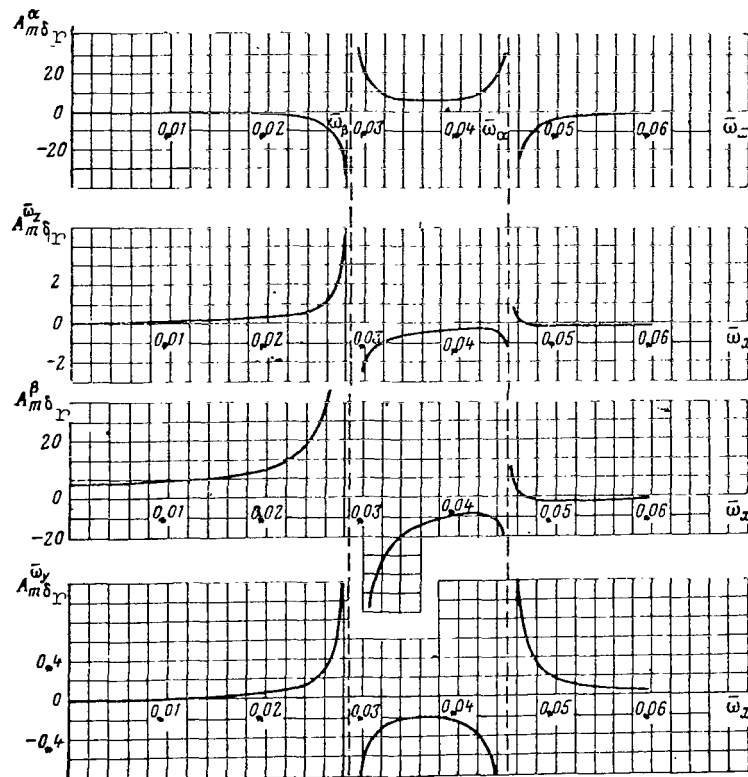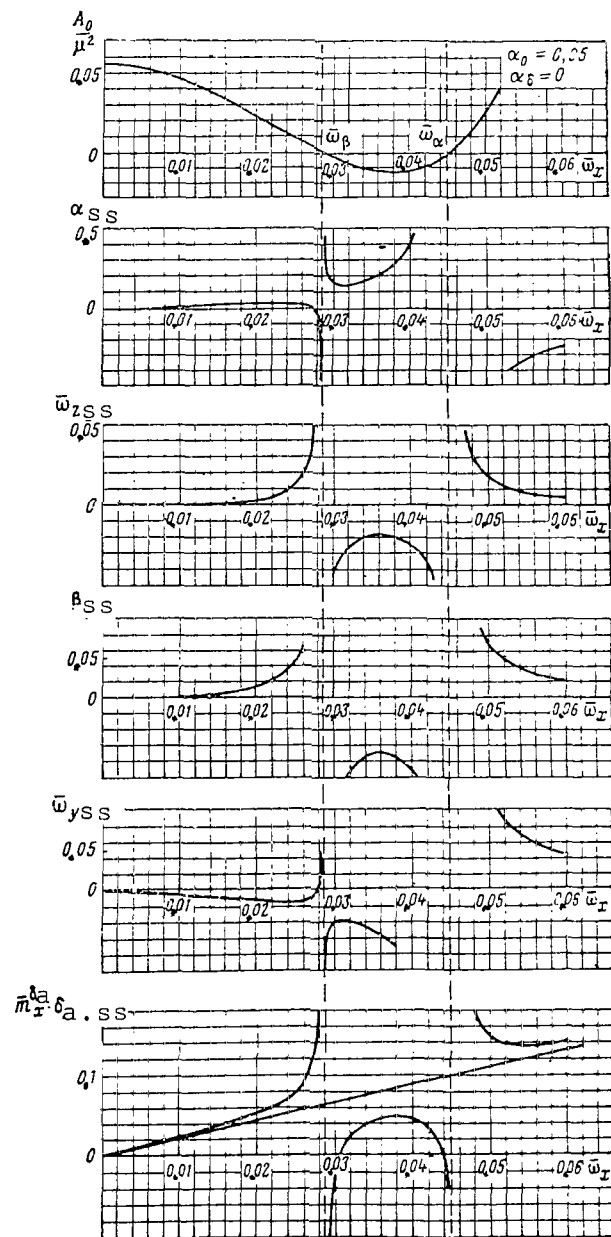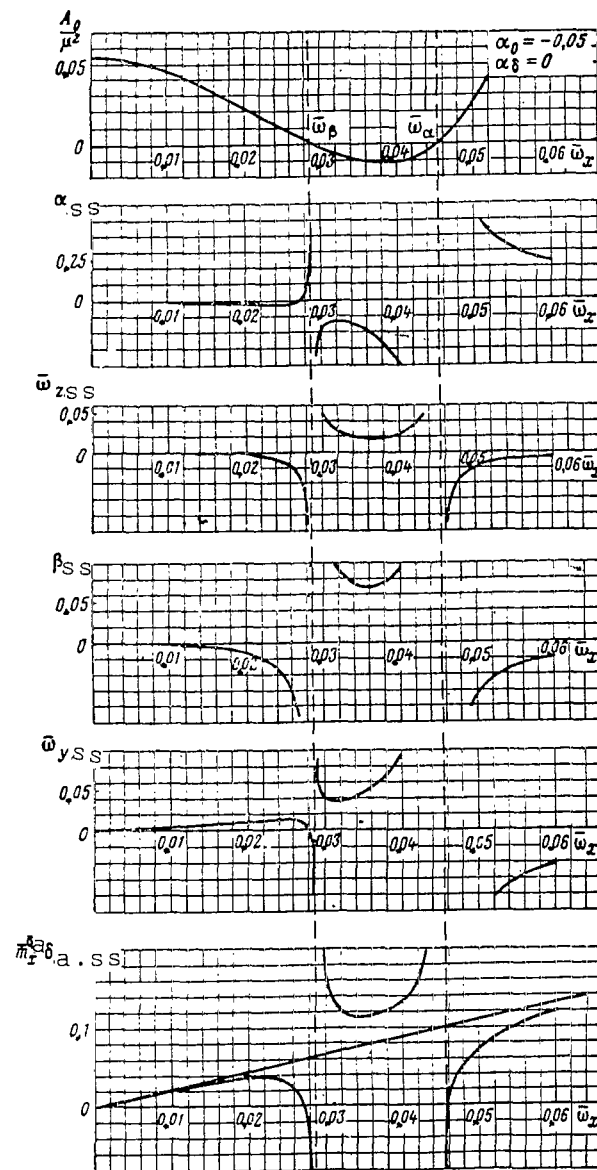

Fig. 3.6.

Fig. 3.7

Fig. 3.8

creating such angular rolling velocities are also infinitely larger if we take into account function (3.13) for an aircraft with a non-zero lateral stability, i.e., when $m_x^\beta \neq 0$ in particular.

Using dependences of the type shown on Figures 3.7 and 3.8 we can find the values of the parameters of motion of an aircraft at the singular points corresponding to a given complex of values characterizing the control of an aircraft (δ_a, δ_e, α_0).

The procedure of finding the variables at the singular points includes the following: Based on the values of $\Delta \overline{m}_z$, $\Delta \overline{m}_y$, α_0, etc., we plot the dependences $\alpha_{ss}(\overline{\omega}_x)$, $\overline{\omega}_{z\,ss}$ $(\overline{\omega}_x)$, $\beta_{ss}(\overline{\omega}_x)$, $\omega_{y\,ss}(\overline{\omega}_x)$ and finally $\Delta \overline{m}_x(\overline{\omega}_x)$. Then according to the value of the aileron deflection (δ_a), for which we must determine the singular points, using the dependence $\Delta \overline{m}_x(\overline{\omega}_x)$, we can find the values $\overline{\omega}_x$, which are values of the angular rolling velocity as well and correspond to the moment of the ailerons $m_x^{\delta_a} \delta_a$. Using the values of $\overline{\omega}_x$ thus found, we can find the remaining parameters of motion of an aircraft at the respective singular points. Let us look, as an example, at all possible singular points – the static systems of motion of an aircraft – under the conditions of horizontal flight with no deflection of the controls:

$$\alpha_0 = 0.05 \,(\mathrm{rad}); \quad \}$$
$$\Delta\delta_e = \delta_r = \delta_a = 0. \quad \}$$

(3.14)

The parameters of motion of an aircraft at the singular points for these conditions of flight can be determined by taking into account the sign of $\overline{\omega}_x$, using Figure 3.7. It is easy to verify that there are five singular points, the parameters of motion in which are tabulated on Table 3.

TABLE 3

No. Parameters	1	2	3	4	5
$\overline{\omega}_x$	0	0.0306	0.0438	−0.0306	−0.0438
α_{ss}	0	0.15	0.9	0.15	0.9
$\omega_{z\,ss}$	0	−0.04	−0.075	−0.04	−0.075
β_{ss}	0	−0.14	−0.18	0.14	0.18
$\omega_{y\,ss}$	0	−0.004	−0.016	0.004	0.016

Let us recall that from the linear theory of aircraft motion we could obtain only one singular point – No. 1. However not all the obtained singular points correspond to the stable motion of an aircraft. In particular in the example studied the only singular point is Point No. 1 which is determined from linear theory. In the general case, several singular points can be found which correspond to stable motion; such systems of controlled motion of an aircraft will be analyzed in detail below.

We can isolate three basic ranges of values of the angular rolling velocity $\bar{\omega}_x$, for each of which the dynamic characteristics of an aircraft have their own singularities. With motion at low angular rolling velocities, when the condition

$$|\bar{\omega}_x| \ll \min(\bar{\omega}_\alpha, \bar{\omega}_\beta),$$ (3.15)

is satisfied [the sign min $(\bar{\omega}_\alpha, \bar{\omega}_\beta)$ denotes the smaller of the two values], the motion of the aircraft is near to that described by the linear differential equations and is analyzed in the majority of papers on dynamics of an aircraft. The second limiting case is the rapid rotation of an aircraft relative to the longitudinal axis with a high angular rolling velocity. Such a motion in its properties is near the rotation of a solid on which the aerodynamic moments do not act since with large values of $\bar{\omega}_x$ the basic role is played by the gyroscopic moments. The angular rolling velocity is assumed high if it satisfies the condition

$$|\bar{\omega}_x| \gg \max(\bar{\omega}_\alpha, \bar{\omega}_\beta),$$ (3.16)

where the sign max $(\bar{\omega}_\alpha, \bar{\omega}_\beta)$ denotes the greater of the two values. This case of aircraft motion will be analyzed in detail in Section 16.

The motion of an aircraft has the most complex characteristics with values of the angular rolling velocities near the critical velocities when the inertial and aerodynamic moments are similar in value. For maneuvering aircraft this range of values in the angular rolling velocity is of the greatest interest.

13. Possible Types of Dependences of the Value of the Moment on the Ailerons Required for Carrying Out a Rolling Maneuver

To obtain some idea as to the qualitative picture of spatial motion of an aircraft, in first order it is necessary to determine the number of type of singular points corresponding to the maneuver being analyzed.

Let us proceed to an investigation of the basic properties and characteristics of the phase pictures corresponding to motion of an aircraft with simultaneous control of rolling and pitching (the most important case in practice), in which the nonlinear character of the motion appears. During rolling maneuvers, control of yawing using the rudder (δ_r) is of least interest and will be studied below rather briefly.

The basic characteristic which determines the number of singular points at given values of the control deflections (see the example examined above), is the function $\Delta \bar{m}_x = f(\bar{\omega}_x, \delta_e \cdot \alpha_0)$. In

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fact, using this function for the known values δ_e, α_0 and δ_a, we can find all the values of the angular rolling velocity $\overline{\omega}_x$ at the singular points which permit determining the remaining parameters of motion at these singular points as well, based on formulas (3.10). Moreover, as will be shown in the next section, the type of the dependence $f(\overline{\omega}_x, \delta_e, \alpha_0)$ permits in any number of cases evaluating more precisely the stability of motion in the vicinity of the singular points found, and discovering the singular points with a prior unstable motion in their vicinity. Taking into account the comments made above, we can study which types of the dependences $\Delta\overline{m}_x = f(\overline{\omega}_x, \delta_e, \alpha_0)$ are possible. The types of functions $f(\overline{\omega}_x, \delta_e, \alpha_0)$ found in such an analysis will be used as the basis of studies of possible types of spatial motion of an aircraft under arbitrary /83 initial conditions and various combinations of control deflections in longitudinal and lateral motions. Let us introduce first instead of the values of the elevator deflection $\Delta\delta_e$ an equivalent value $\Delta\alpha_b$ which represents the increase in the angle of attack in isolated longitudinal motion that is determined by the approximation relationship

$$\Delta\alpha_b \cong -\frac{\overline{m}_z^{\delta_e}\,\Delta\delta_e}{\overline{m}_z^{\alpha}}.$$

(3.17)

Taking into account Equation (3.17), the maneuvers are characterized by the size of the aileron deflection and by the two components of the total angle of attack α_0 and $\Delta\alpha$. By definition, the angle of attack α_0 is the angle between the vector of the flying speed (when $\beta = 0$) and the major inertial axis of the aircraft OX_1 during horizontal flight, i.e., when $\overline{\omega}_{z0} = 0$. The angle $\Delta\alpha$ is equal to the increase in the angle of attack of the aircraft when the elevator is deflected by a value $\Delta\delta_e$ relative to the trim position in horizontal flight. The appearance of the angle $\Delta\alpha$ is accompanied by the development of an angular pitching velocity $\overline{\omega}_{z0} \neq 0$ which from the very beginning of the maneuver "leads" the vector of the total angular velocity of the aircraft away from the axis OX_1. This produces a slight difference in the influence of the angles α_0 and $\Delta\alpha$ on the motion of the aircraft. Let us recall that the introduction into the equations of motion of the angle of attack α_0 is associated with the necessity of taking into account the influence of gravity on the trim of an aircraft.

Since the projections of the gravitational forces onto the body axes OY_1Z_1, when an aircraft rolls in a turn, become almost periodic functions of time and condition of flight when α_0 with $\overline{\omega}_z = 0$ are disrupted, such a division of angles, strictly speaking, is substantial only in evaluating the beginning of the rolling maneuver. In analyzing lengthy rolling maneuvers we can approximately assume the angles α_0 and $\Delta\alpha_b$ to be equivalent. Somewhat later we shall look at this problem in greater detail.

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Let us proceed to an anlysis of the possible types of the dependence $\Delta m_x(\overline{\omega}_x, \alpha_0, \Delta\alpha_b)$. The value of the rolling moment required for trim of the aircraft during rotation at an angular velocity $\overline{\omega}_x$ is determined from the algebraic relationship obtained by equating the derivative $\overline{\omega}_x$ to zero:

$$\Delta\overline{m}_x = \overline{m}_x^{\delta_a} \cdot \overline{\delta}_a = -\overline{m}_x^{\omega_x} \cdot \overline{\omega}_x - \overline{m}_x^\beta \cdot \beta_{ss} + C\mu\overline{\omega}_{z\,ss} \cdot \overline{\omega}_{y\,ss} \tag{3.18}$$

In Equation (3.18) for simplification of computation only the basic terms are retained, in particular the term $\overline{m}_x^{\omega_y}\omega_{y\,ss}$, etc. is omitted. As computations show the term $C\mu\overline{\omega}_{z\,ss} \cdot \overline{\omega}_{y\,ss}$ exerts an influence on the character of the change in the function $\Delta m_x(\overline{\omega}_x, \alpha_0, \Delta\alpha_b)$ only in the immediate vicinity of the critical angular rolling velocities where this term becomes determinant. With angular rolling velocities not very close to the critical, its influence is not substantial and in qualitative evaluations may not be taken into account. Due to this fact the problem of investigating the dependence $\Delta\overline{m}_x(\overline{\omega}_x, \alpha_0, \Delta\alpha_b)$ can be divided into two parts: investigation at angular rolling velocities $(\overline{\omega}_x)$ different from the critical, and investigation at values of $\overline{\omega}_x$ in the immediate vicinity of the critical rolling velocity. /84

Let us set into relationship (3.18) the product of the angular velocities equal to zero. Such a simplification will permit clarifying the basic laws governing the change in the function $\Delta\overline{m}_x(\overline{\omega}_x, \alpha_0, \Delta\alpha_b)$, which determine the controlled motion of an aircraft.

Let us set into Equation (3.18) an expression for β_{ss}, which is determined by using Equation (3.10) and Table 2.

$$\Delta\overline{m}_x = -\frac{\overline{\omega}_x\mu^2}{A_0}\left\{\frac{\overline{m}_x^{\omega_x}\cdot A_0}{\mu^2} + \overline{m}_x^\beta\left[-\overline{m}_{zb}^\gamma\Delta\alpha_b\left(B\frac{c_y^\alpha}{2} - \overline{m}_y^{\omega_y}\right) + \right.\right.$$
$$\left.\left. + \alpha_0\left(\frac{c_y^\alpha\,\overline{m}_{zb}^{\omega_z}}{2\mu}\,\overline{m}_y^{\omega_y} + \overline{m}_{zb}^\alpha\overline{m}_y^{\omega_y} + \mu AB\overline{\omega}_x^2\frac{c_y^\alpha}{2}\right)\right]\right\} \tag{3.19}$$

If we make the necessary conversions and group the terms by identical powers of $\overline{\omega}_x$, we can reduce formula (3.19) to a form that is convenient for analysis.

$$\Delta\overline{m}_x = -\frac{\mu^2\overline{\omega}_x\,\overline{m}_x^{\omega_x}}{A_0}\cdot C_1(\overline{\omega}_x). \tag{3.20}$$

In Equation (3.20) the following definitions are used:

$$C_1(\overline{\omega}_x) = D_2\overline{\omega}_x^4 + D_1\overline{\omega}_x^2 + D_0,$$

(3.21)

where

$$D_2 = AB\mu^2;$$

$$D_1 = \left(\overline{m}_{zb}^\alpha + \frac{c_y^\alpha \overline{m}_z^{\omega_z}}{2\mu}b\right)B\mu + \left(\overline{m}_y^\beta - \frac{c_z^\beta \overline{m}_y^{\omega_y}}{2\mu}\right)A\mu +$$

$$+ \left(-A\frac{c_z^\beta}{2} - \overline{m}_{zb}^{\omega_z}\right)\left(B\frac{c_y^\alpha}{2} - \overline{m}_y^{\omega_y}\right) + \mu\frac{\overline{m}_x^\beta}{\overline{m}_x^{\omega_x}}AB\frac{c_y^\alpha}{2}\alpha_0;$$

$$D_0 = \left(\overline{m}_{zb}^\alpha + \frac{c_y^\alpha \overline{m}_z^{\omega_z}}{2\mu}b\right)\left(\overline{m}_y^\beta - \frac{c_z^\beta \overline{m}_y^{\omega_y}}{2\mu}\right) -$$

$$- \frac{\overline{m}_x^\beta}{\overline{m}_x^{\omega_x}}\overline{m}_{zb}^\alpha\left(\frac{c_y^\alpha}{2}B\Delta\alpha_b - \overline{m}_y^{\omega_y}\cdot\Delta\alpha_b - \overline{m}_y^{\omega_y}\alpha_0 - \frac{c_y^\alpha \overline{m}_{zb}^{\omega_z}}{2\mu\overline{m}_{zb}^\alpha}\right).$$

(3.22)

Since the parameters of control in the longitudinal plane (α_0 and $\Delta\alpha_b$) do not influence the function $A_0(\overline{\omega}_x)$, to determine the dependence $\Delta\overline{m}_x(\overline{\omega}_x, \alpha_0, \Delta\alpha_b)$ it is necessary and sufficient to investigate the function $C_1(\overline{\omega}_x)$, which stands in the numerator. From formula (3.21) it follows that the function $C_1(\overline{\omega}_x)$ is an $\overline{\omega}_x^2$ parabola and consequently is symmetric relative to the axis $\overline{\omega}_x = 0$. When $|\overline{\omega}_x| \to \infty$, regardless of the parameters of control, the quantity $C_1 \to \infty$. From this result it follows that the type of parabola $C_1(\overline{\omega}_x^2)$ can be fully determined by the values of the zeros since the direction of its curvature is known.

Let us proceed to a more detailed analysis of the properties of the function $C_1(\overline{\omega}_x^2)$. Let us determine the possible types of curves $C_1(\overline{\omega}_x^2)$ and the functions $\Delta\overline{m}_x(\overline{\omega}_x, \alpha_0, \Delta\alpha_b)$ which correspond to them. Let us look first of all at the behavior of the function $C_1(\overline{\omega}_x^2)$ in the vicinity of the point $\overline{\omega}_x = 0$. Two cases are possible: $C_1(0) > 0$ and $C_1(0) < 0$. In satisfying the condition

$$C_1(0) > 0$$

(3.23)

the function $C_1(\overline{\omega}_x)$ may have either two positive roots λ_1 and λ_2 or no root at all or in the basic case when the curve $C_1(\overline{\omega}_x)$ is tangent to the abscissa axis - one root (Fig. 3.9). In that case when $C_1(0) \le 0$ the function $C_1(\overline{\omega}_x)$ has only one positive root. We can express condition (3.23) through the parameters of the aircraft as a characteristic of the motion in the longitudinal plane. To simplify

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the writing let us omit small values such as

$$\frac{c_y^\alpha \, \overline{m}_z^{\overline{\omega}z} b}{2\mu} \quad \text{and} \quad \frac{c_z^3 \, \overline{m}_y^{\overline{\omega}y}}{2\mu}.$$

Let us obtain

$$\alpha_0 - \frac{\left(B\frac{c_y^\alpha}{2} - \overline{m}_y^{\overline{\omega}y}\right)}{\overline{m}_y^{\overline{\omega}y}}\Delta\alpha_b + \frac{\overline{m}_x^{\overline{\omega}x}\,\overline{m}_y^3}{\overline{m}_x^\beta\,\overline{m}_y^{\overline{\omega}y}} \geqslant 0. \tag{3.24}$$

Investigation of the different types of functions $C_1(\overline{\omega}_x)$ can be conveniently carried out on the planes of the parameters α_0, $\Delta\alpha_b$. The straight line which is described by Equation (3.24) (Line 1 on Fig. 3.10), divides the plane of the parameters α_0, $\Delta\alpha_b$ into two regions, in one of which the function $C_1(\overline{\omega}_x)$ has either two or no zeros, and in the others has one zero.

Now let us define on the plane $(\alpha_0, \Delta\alpha_b)$ a region in which the function $C_1(\overline{\omega}_x)$ does not have any zeros, i.e., a region where the function C_1 is constant in sign. The condition of constant signs of the function $C_1(\overline{\omega}_x)$ is that the subroot expressions (discriminant) in the formula for the roots of the polynomial $C_1(\overline{\omega}_x)$ be negative:

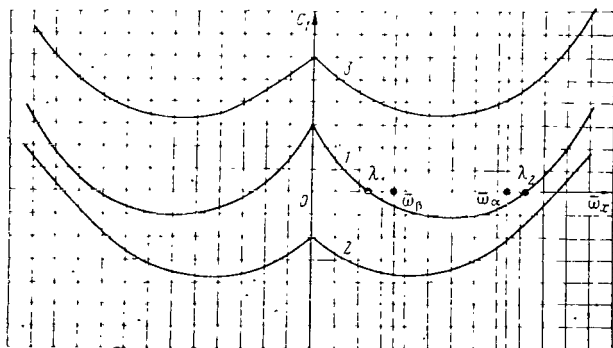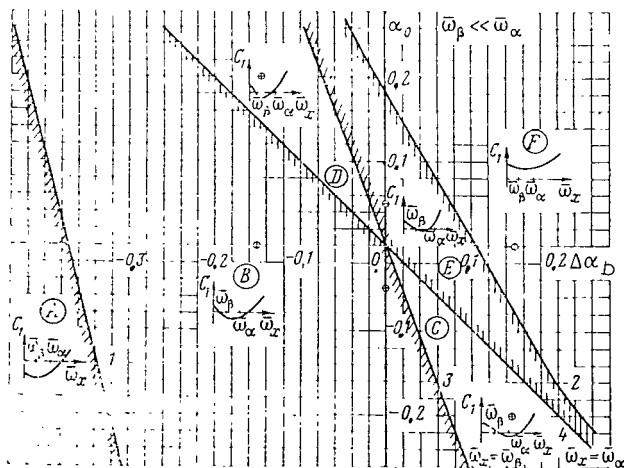$$D_1^2 - 4D_2D_0 \leqslant 0. \tag{3.25}$$

Fig. 3.9

Fig. 3.10.

If we substitute into Formula (3.25) expressions for D_2, D_1 and D_0 through the aerodynamic coefficients and carry out the necessary conversions we find the approximate condition for constant signs of the function $C_1(\bar\omega_x)$ in the form of an inequality

$$
\Delta\alpha_6 + \frac{\bar{m}_x^\beta}{4\,\bar{m}_x^{\bar\omega_x}} \cdot \frac{AB\left(\dfrac{c_y^\alpha}{2}\right)^2}{\bar{m}_z^\alpha{}_b\left(\dfrac{c_y^\alpha}{2}B - \bar{m}_y^{\bar\omega_y}\right)}\,\alpha_0^2 +
$$

$$
+ \frac{\left[\bar{m}_z^\alpha{}_b B\mu + \bar{m}_y^\beta A\mu + \left(-A\dfrac{c_z^\beta}{2} - \bar{m}_z^{\bar\omega_z}{}_b\right)\left(B\dfrac{c_y^\alpha}{2} - \bar{m}_y^{\bar\omega_y}\right)\right]\dfrac{c_y^\alpha}{2\mu} - 2\bar{m}_z^\alpha{}_b\,\bar{m}_y^{\bar\omega_y}}{4\,\bar{m}_z^\alpha{}_b\left(B\dfrac{c_y^\alpha}{2} - \bar{m}_y^{\bar\omega_y}\right)}\,\alpha_0 +
$$

$$
+ \frac{\left[\bar{m}_z^\alpha{}_b B\mu + \bar{m}_y^\beta A\mu + \left(-A\dfrac{c_z^\beta}{2} - \bar{m}_z^{\bar\omega_z}{}_b\right)\left(B\dfrac{c_y^\alpha}{2} - \bar{m}_y^{\bar\omega_y}\right)\right]^2 - 4AB\mu^2\,\bar{m}_z^\alpha{}_b\,\bar{m}_y^\beta}{4AB\mu^2\,\dfrac{\bar{m}_x^\beta}{\bar{m}_x^{\bar\omega_x}}\,\bar{m}_z^\alpha{}_b\left(B\dfrac{c_y^\alpha}{2} - \bar{m}_y^{\bar\omega_y}\right)} \geqslant 0.
$$

(3.26)

For the small values of interest to us for the trim angle α_0 the second term in expression (3.26) is small and can be dropped. In this case the condition of constant signs of the function $C_1(\bar\omega_x)$ can be simplified. The straight line obtained from Equation (3.26) is shown on Figure 3.10 by the number "2". In the region between the two straight lines which are described by Equations (3.24) and (3.26), the function $C_1(\bar\omega_x)$ has two zeros.

Of special interest is the position of the zeros λ_1, λ_2 of the function $C_1(\bar\omega_x)$ with respect to the zeros of the free term A_0, since the form of the function $\Delta\bar{m}_x(\bar\omega_x, \alpha_0, \Delta\alpha_b)$ which is proportional to the ratio of $C_1(\bar\omega_x)$ to A_0 depends on their mutual location. The equation of the curves $(\alpha_0, \Delta\alpha_b)$, on which the zeros of the function $C_1(\bar\omega_x)$ and the zeros of A_0 coincide, can be found after substituting into expression (3.21) for the function $C_1(\bar\omega_x)$ the values of $\bar\omega_x$ equal to the zeros of A_0:

$$
\bar\omega_x = \bar\omega_\alpha;
$$

(3.27)

$$
\bar\omega_x = \bar\omega_\beta.
$$

(3.28)

If we substitute (3.27) and (3.28) into Formula (3.21), we find two equations for the boundaries of such regions:

$$
AB\mu^2\bar\omega_\alpha^4 + \bar\omega_\alpha^2 \cdot D_1 + D_0 = 0;
$$

(3.29)

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$$AB\mu^2\overline{\omega}_\beta^{-4} + \overline{\omega}_\beta^2 \cdot D_1 + D_0 = 0. \tag{3.30}$$

Expressions (3.29) and (3.30) are valid in all cases when the $\underline{/88}$
critical angular rolling velocities $\overline{\omega}_\alpha$ and $\overline{\omega}_\beta$ exist, i.e., A_0 has
zeros. Using the relationship which appears in the definition for
$\overline{\omega}_\alpha$ and $\overline{\omega}_\beta$

$$A_0(\overline{\omega}_\alpha) = A_0(\overline{\omega}_\beta) = 0, \tag{3.31}$$

expressions (3.29) and (3.30) can be simplified. After carrying
out the necessary calculations we find

$$\left(\overline{\omega}_\alpha^2 A\mu B \frac{c_y^\alpha}{2} + \overline{m}_{z\,b}^\alpha \overline{m}_y^{\overline{\omega}y}\right)\alpha_0 - \overline{m}_{z\,b}^\alpha\left(\frac{c_y^\alpha}{2}B - \overline{m}_y^{\overline{\omega}y}\right)\Delta\alpha_b = 0; \tag{3.32}$$

$$\left(\overline{\omega}_\beta^2 A\mu B \frac{c_y^\alpha}{2} + \overline{m}_{z\,b}^\alpha \overline{m}_y^{\overline{\omega}y}\right)\alpha_0 - \overline{m}_{z\,b}^\alpha\left(\frac{c_y^\alpha}{2}B - \overline{m}_y^{\overline{\omega}y}\right)\Delta\alpha_b = 0. \tag{3.33}$$

From expressions (3.32) and (3.33) it follows that the lines which
divide the regions that differ in the different mutual position of
zeros of the functions $C_1(\overline{\omega}_x)$ and $A_0(\overline{\omega}_x)$, are straight lines which
pass through the origin $\alpha_0 = \Delta\alpha_b = 0$ (see Fig. 3.10, Curves 3 and
4).

In the case which is of the most practical interest in analyzing
the dynamics of an aircraft with small damping, expressions (3.32)
and (3.33) can be simplified even more. To do so it is necessary
to use the approximate formulas for $\overline{\omega}_\alpha$ and $\overline{\omega}_\beta$:

$$\omega_\alpha^2 \approx -\frac{m_{z\,b}^\gamma}{A\mu}; \quad \omega_\beta^2 \approx -\frac{m_y^\beta}{B\mu}. \tag{3.34}$$

Using the expressions for the critical angular velocities (3.34)
with the aid of Formulas (3.32) and (3.33) we find approximate
equations of the boundaries for the regions of dividing the planes
α_0, $\Delta\alpha_b$:

$$\alpha_0 + \Delta\alpha_b = 0; \tag{3.35}$$

$$\frac{\left[\left(\dfrac{\overline{m}_y^\beta A}{\overline{m}_{z\,b}^\alpha B}\right) - \dfrac{2\overline{m}_y^{\overline{\omega}y}}{c_y^\alpha B}\right]}{1 - \dfrac{2\overline{m}_y^{\overline{\omega}y}}{c_y^\alpha B}}\alpha_0 + \Delta\alpha_b = 0. \tag{3.36}$$

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The analysis which we carried out of the function $C_1(\bar{\omega}_x)$ permitted us to clarify its basic properties and to divide the plane $(\alpha_0, \Delta\alpha_b)$ into regions with different characteristics of behavior of the curve $C_1(\bar{\omega}_x)$ with respect to $A_0(\bar{\omega}_x)$. Examples of such a division for the various relationships between the critical rolling velocities $\bar{\omega}_\alpha$ and $\bar{\omega}_\beta$ are plotted on Figures 3.10 and 3.11. From comparison of these figures, it is obvious that the mutual position of the regions with different characteristics of change in $C_1(\bar{\omega}_x)$ is independent of the relationship between the critical angular rolling velocities. This is due to the fact that simultaneously with the change in the mutual position of Curves 3 and 4 the mutual position of the critical rolling velocities $\bar{\omega}_\alpha$ and $\bar{\omega}_\beta$ were changed.
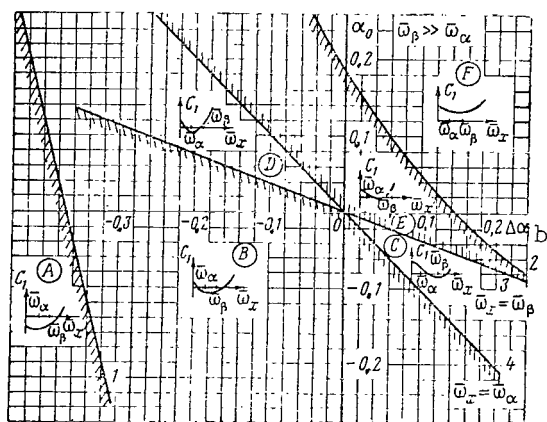

Fig. 3.11

Let us recall that the function $\Delta\bar{m}_x(\bar{\omega}_x)$ is represented in the form of Expression (3.20). Taking this into account and using Figures 3.10 and 3.11, we can define six basic types of the functions $\Delta\bar{m}_x = (\bar{\omega}_x, \alpha_0, \Delta\alpha_b)$, which are illustrated on Figure 3.12. Of greatest interest is the character of change of these functions in the regions of the angular rolling velocities less than the first and greater than the second critical. The dependence of the dynamic characteristics of an aircraft on the form of the function $\Delta\bar{m}_x(\bar{\omega}_x)$ will be studied in greater detail below; however, it is feasible to make several comments even at this stage of the study. Even a rapid glance at the curve of the static solutions shows that in the range of the angular rolling velocities less than the first critical, the function $\Delta\bar{m}_x(\bar{\omega}_x)$ tends either toward $+\infty$ or toward $-\infty$, hence in particular it follows that in the second case with deflection of the ailerons by a value which is greater than a certain value, the continuous relationship between the value $\Delta\bar{m}_x$ and the angular rolling velocity is disrupted. In this case the aircraft may develop angular velocities greater than the second critical ($\omega_{2\ crit}$). From Figure 3.12 it follows that such properties are inherent in the motion of an aircraft in Cases C and E.

In the range of angular rolling velocities which exceeds the second critical, of the most practical interest is the question of the existence of an intersection of the curve of the static function $\Delta\bar{m}_x(\bar{\omega}_x)$ with the axis $\Delta\bar{m}_x = 0$. The existence of such an intersection (maneuvers of type A, B, C) indicates the presence of a singular point corresponding to the motion of an aircraft with angular rolling velocity $\bar{\omega}_x > \bar{\omega}_{2\ crit}$ with no deflection of the ailerons. It is obvious that investigation of such cases is of the most practical interest since the entry of an aircraft into such conditions of

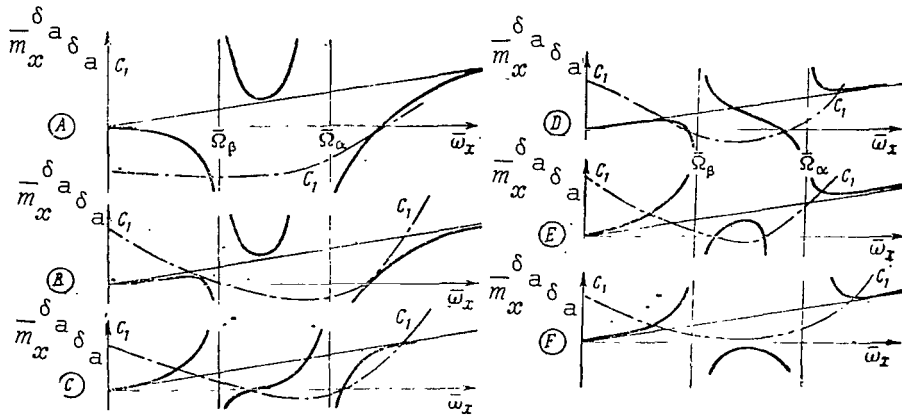motion means simply the loss of controllability by the ailerons.

Fig. 3.12

Let us proceed to an investigation of the behavior of the curves of the static functions $\Delta \overline{m}_x(\overline{\omega}_x, \alpha_0, \Delta\alpha_b)$ in the immediate vicinity of the singular point. The results obtained above are basic for analyzing the dynamics of an aircraft, and investigation of the static curves in the vicinity of the singular points is mainly of theoretical interest. As noted above, in those cases when the inertial moments of an aircraft J_y and J_z are not mutually equal, the character of the change in the function $\Delta m_x(\omega_x)$ in the vicinity of the singular points is determined by the term $C\overline{\omega}_{y\text{ss}}\omega_{z\text{ss}}\mu$ in Expression (3.18). The inequality of the inertial moments J_y and J_z are characteristic for aircraft of ordinary design. In fact if the inertial moment of an aircraft J_x is determined mainly by the mass which is distributed in the wings then as can be shown by using the definitions in Figure 3.13 the following approximate relationship is valid:

$$J_z \cong J_y - J_x.$$ (3.37)

In fact, the expressions for the inertial moments may be approximately written in the following form:

$$J_x \simeq 2M_2 r_z^2;$$
$$J_z \simeq 2M_1 r_x^2;$$
$$J_y \doteq 2(M_1 r_x^2 + M_2 r_z^2),$$

whence it is easy to obtain Equation (3.37). In turn, from Equation (3.37) it follows that the quantity C is approximately equal to unity with a minus sign.

Using the relationships in Table 2 and analyzing as the parameters α_0 and $\Delta\alpha_b$, we can write an expression for the product $(\overline{\omega}_{yss} \cdot \overline{\omega}_{zss})$:

$$\overline{\omega}_{yss}\overline{\omega}_{zss} = \frac{\mu^4}{A_0^2}(C_{11}\Delta\alpha_b + C_{12}\alpha_0)(C_{21}\Delta\alpha_b + C_{22}\alpha_0),$$

(3.38)

where

$$
\left.
\begin{aligned}
C_{11} &= \frac{\overline{\omega}_x}{\overline{m}_z^\alpha}\left(\mu B\overline{\omega}_x^2 + \overline{m}_y^\beta - B\frac{C_y^\alpha C_z^\beta}{4\mu}\right); \\[2mm]
C_{12} &= -\overline{\omega}_x\left(\overline{m}_y^\beta \overline{m}_{zb}^{\alpha}\mu + \mu B\overline{\omega}_x^2 \overline{m}_{zb}^\alpha + \frac{c_y^\alpha \overline{m}_z^{\omega_z}}{2\mu}b - \overline{m}_y^\beta\right); \\[2mm]
C_{21} &= \frac{1}{\overline{m}_z^\alpha}\left(\frac{c_y^\alpha c_z^\beta \overline{m}_y^{\omega_y}}{4\mu} - \overline{\omega}_x^2 \overline{m}_y^{\omega_y} - \frac{c_y^\alpha \overline{m}_y^\beta}{2\mu}\right); \\[2mm]
C_{22} &= \overline{\omega}_x^2\left(-\overline{m}_y^{\omega_y}\overline{m}_{zb}^\alpha + A\overline{m}_y^\beta\frac{c_y^\alpha}{2}\right).
\end{aligned}
\right\}
$$

(3.39)
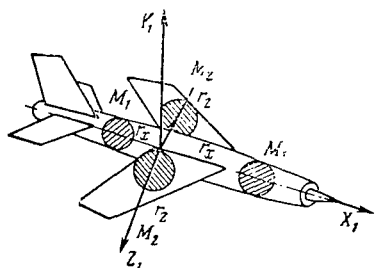
Fig. 3.13

From Formula (3.38) it follows that the function $\Delta\overline{m}_x(\overline{\omega}_x, \alpha_0, \Delta\alpha_b)$ with critical angular rolling velocities has a terminal of second order (and the denominator A_0 enters in the second power). With the passage of $\overline{\omega}_x$ through the values of the critical angular rolling velocities the sign of the function $\Delta\overline{m}_x(\overline{\omega}_x, \alpha_0, \Delta\alpha_b)$ is not changed since the denominator (A_0^2) is always positive. In this respect the sign of $\Delta\overline{m}_x(\overline{\omega}_x)$ in the vicinity of the critical angular velocity is determined only by the sign of the numerator in Formula (3.20). With identical signs of the brackets, the function $\Delta m_x(\overline{\omega}_x, \alpha_0, \Delta\alpha_b)$, when $\overline{\omega}_x \to \overline{\omega}_x$ crit tends toward $(+\infty)$, and with different signs to $(-\infty)$. The boundaries of the ranges in this case are straight lines which are determined by the equations

$$
\left.
\begin{aligned}
C_{11}\Delta\alpha_b + C_{12}\alpha_0 &= 0; \\
C_{12}\Delta\alpha_b + C_{22}\alpha_0 &= 0.
\end{aligned}
\right\}
$$

(3.40)

Direct proof is easy to find that in the case when $\overline{\omega}_\alpha \gg \overline{\omega}_\beta$, all the coefficients C_{11}, C_{12}, C_{21}, C_{22} are positive. The division of the plane α_0, $\Delta\alpha_b$ into regions where the static functions to to $+\infty$ (first type) and to $-\infty$ (second type) is clear from Figure 3.14, a and be. The straight lines are plotted for the parameters of an

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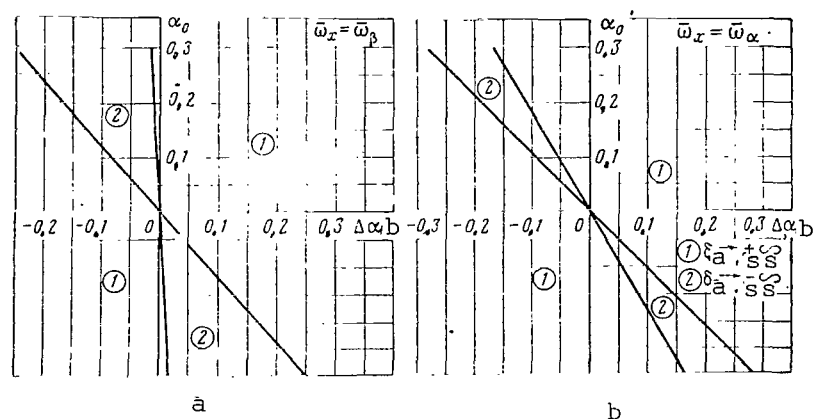aircraft when $\bar{\omega}_\alpha \gg \bar{\omega}_\beta$.

Fig. 3.14

From the analysis carried out it follows that four different types of changes are possible in the function $\Delta \bar{m}_x (\bar{\omega}_x, \alpha_0, \Delta\alpha_b)$ in the immediate vicinity of the critical angular velocities that differ in the character of the change $\Delta \bar{m}_x$.

Table 4 gives a summation of the possible groups of types of functions $\Delta \bar{m}_x (\bar{\omega}_x)$ in the vicinities of the critical velocities.

In this case the number "1" denotes growth in the function $\Delta \bar{m}_x$ in the positive direction ($+\infty$) and the number "2" in the negative direction ($-\infty$).

Depending on the group of parameters α_0 and $\Delta\alpha_b$ for which the rolling maneuver is carried out, one of the four types of changes in the function $\Delta \bar{m}_x (\bar{\omega}_x)$ is possible in the vicinity of the critical velocities. The proper division of the plane (α_0, $\Delta\alpha_b$) can be obtained by using Figure 3.14, a and b. An example of such division is shown on Figure 3.15.

Thus the investigation which we carried out shows that in the general case when there are two varying parameters of the longitudinal control $\Delta\alpha_b$ and α_0, six possible types of change in the function $\Delta \bar{m}_x (\bar{\omega}_x)$ are possible, in each of which in the immediate vicinity of the critical velocities $\bar{\omega}_x$ the function can be still one of four types. However, as computations show for practical problems there is no necessity for analyzing the behavior of the curve $\Delta \bar{m}_x (\bar{\omega}_x)$ in the immediate vicinity of the critical rolling velocities.

TABLE 4

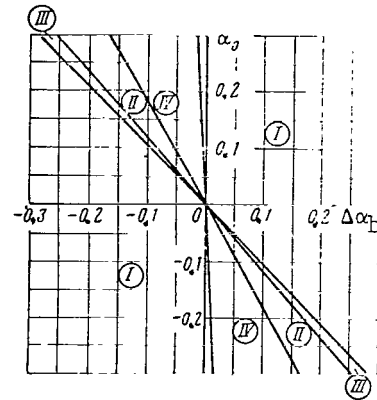| Type	$\bar{\omega}_x = \bar{\omega}_\beta$	$\bar{\omega}_x = \bar{\omega}_\alpha$
I	1	1
II	2	2
III	1	2
IV	2	1

Fig. 3.15

14. Analysis of the Stability of Motion in the Vicinity of the Singular Point

As noted in Section 11, the phase trajectories possess the property that they do not intersect anywhere, with the exception of the "singular" points or points of "rest" of the system. In this case the phase trajectories may either "enter into" the singular point (in this case the singular point corresponds to a state of stable equilibrium), pass alongside, or "leave from" it (unstable equilibrium). A detailed investigation of the motion of an aircraft in phase space with the location of the phase trajectories at all points of this space is quite complex and there is no necessity for so doing. Of considerably greater interest is the simpler problem of finding the position of the singular points in phase space and the investigation of the motion of an aircraft in their vicinity.

The problem of determining the type of singular point is reduced to an analysis of the motion in its vicinity, i.e., to an investigation of the motion "in the small" and therefore is extremely close to the problem of analyzing the stability of motion near the singular point. In this case the knowledge and use of the criteria of stability or instability of motion permits determining to which type the singular point under investigation belongs, and also as to whether the motion in its vicinity can be practically realized. Unfortunately in the form that is used for analytical investigations all the criteria of stability of motion cannot be obtained, however one of the most important criteria, i.e., the criteria of the aperiodic stability, can be reduced to a rather simple and clear form.   /94

In analyzing the motion of an aircraft in a certain small vicinity of the singular point, Equations (1.33) can be simplified by converting them to equations in variations relative to the parameters corresponding to the motion at the singular point of the system of equations of motion. Let us write, as is usually done in linearizing equations, the parameters of motion of the aircraft in

the form

$$
\left.
\begin{aligned}
\alpha &= (c_0 + \alpha_{ss}) + \Delta\alpha; \\
\bar{\omega}_z &= \bar{\omega}_{z\,ss} + \Delta\bar{\omega}_z; \\
\beta &= \beta_{ss} + \Delta\beta; \\
\bar{\omega}_y &= \bar{\omega}_{y\,ss} + \Delta\bar{\omega}_y; \\
\bar{\omega}_x &= \bar{\omega}_{x0} + \Delta\bar{\omega}_x.
\end{aligned}
\right\}
\tag{3.41}
$$

Carrying out the usual procedure of linearization we find the system of uniform equations in the variations (for brevity of description the sign of variation Δ in the equations is omitted):

$$
\left.
\begin{aligned}
\alpha' &+ \frac{c_y^a}{2}\,\alpha - \mu\bar{\omega}_z + \mu\beta\bar{\omega}_{x0} + \mu\beta_{ss}\bar{\omega}_x = 0; \\
\bar{\omega}_z' &- \bar{m}_{zb}^{\alpha}\,\alpha - \bar{m}_{zb}^{\bar{\omega}_z}\bar{\omega}_z + A\mu\bar{\omega}_{x0}\cdot\bar{\omega}_y + A\mu\bar{\omega}_{y\,ss}\bar{\omega}_x = 0; \\
\beta' &- \mu\bar{\omega}_y - \frac{c_z^{\beta}}{2}\,\beta - \mu(\alpha_0 + \alpha_{cT})\bar{\omega}_x - \mu\bar{\omega}_{x0}\alpha = 0; \\
\bar{\omega}_y' &- \bar{m}_y^{\beta}\beta - \bar{m}_y^{\bar{\omega}_y}\cdot\bar{\omega}_y - \mu B\bar{\omega}_{x0}\cdot\bar{\omega}_z - \mu B\bar{\omega}_{z\,ss}\bar{\omega}_x = 0; \\
\bar{\omega}_x' &- \bar{m}_x^{\bar{\omega}_x}\bar{\omega}_x - \mu C\bar{\omega}_{y\,ss}\bar{\omega}_z - \bar{m}_x^{\beta}\beta - \left(\mu C\bar{\omega}_{z\,ss} + \bar{m}_x^{\bar{\omega}_y}\right)\bar{\omega}_y = 0.
\end{aligned}
\right\}
\tag{3.42}
$$

To define the conditions of stability, it is necessary to look at the characteristic equation of the system of Equations (3.42), which in the general case can be written in the form

$$
\lambda^5 + B_4\lambda^4 + B_3\lambda^3 + B_2\lambda^2 + B_1\lambda + B_0 = 0.
\tag{3.43}
$$

The expressions for the coefficient B_4, B_3, B_2, B_1, B_0 through the aerodynamic and inertial characteristics of the aircraft are quite awkward and cited in Table 5.

Before we proceed to an analysis of the Equations (3.43) we must make several comments. The problem which is examined in this section is quite near to that studied in Chapter II. The difference consists of the fact that the stability of motion of an aircraft when $\bar{\omega}_x \equiv$ const was analyzed earlier which in particular corresponds to the equations C, $\bar{m}_x^{\beta}$, $\bar{m}_x^{\omega y}$ being equated to zero in Equations (3.42), etc., (of all coefficients in the equation of the rolling moments other than $\bar{m}_x^{\omega x}$ etc.). In the present section we look at the stability of motion of an aircraft by taking into account the fact that in the general case the angular rolling velocity [because, e.g., of the presence in the aircraft of lateral stability ($\bar{m}_x^{\beta} \neq 0$)]

TABLE 5

B_4	$A_3 - \overline{m}_x^{\overline{\omega}} x$
B_3	$A_2 - A_3 \overline{m}_x^{\overline{\omega}} x - \overline{m}_x^{\beta} \mu \, (\alpha_{\overline{S} \, \overline{S}} + \alpha_0) + \mu \, (\mu C \overline{\omega}_z \,_{S \, S} - \overline{m}_x^{\overline{\omega}} y) \cdot \{ - B \overline{\omega}_z S \, S + \mu^2 C \overline{\omega}_y \,_S \, S \{ A \overline{\omega}_y \, SS \}$
B_2	$A_1 - A_2 \overline{m}_x^{\overline{\omega}} x - \overline{m}_x^{\beta} \mu \, [- \beta_{SS} \mu \Omega + (\alpha_{S \, S} + \alpha_0) \times$ $\times \left(\dfrac{c_y^{\alpha}}{2} - \overline{m}_z^{\overline{\omega}} _b - \overline{m}_y^{\overline{\omega}} y \right) + B \, \overline{\omega}_z \,_{S \, S} \mu] + \mu \, (\mu C \overline{\omega}_z \,_{S \, S} -$ $- \overline{m}_x^{\overline{\omega}} y) \left[A \overline{\omega}_y \,_{S \, S} B \Omega + (\alpha_{S \, S} + \alpha_0) \, (- \overline{m}_y^{\beta}) - \right.$ $- B \overline{\omega}_z \,_S \,_S \left(\dfrac{c_y^{\alpha}}{2} - \dfrac{c_z^{\beta}}{2} - \overline{m}_{z \, b}^{\overline{\omega}} \right)] + \mu^2 C \overline{\omega}_y \,_{SS} [+ \beta_{cT} \cdot \overline{m}_z^{\alpha} \,_b +$ $+ A \overline{\omega}_y \,_S \,_S \left(\dfrac{c_y^{\alpha}}{2} - \dfrac{c_z^{\beta}}{2} - \overline{m}_y^{\overline{\omega}} y \right) + B \overline{\omega}_z \,_S \,_S A \mu \Omega]$
B_1	$A_0 - A_1 \overline{m}_x^{\overline{\omega}} x - \overline{m}_x^{\beta} \mu \left\{ - \beta_S \,_S \left[- \mu \Omega \left(\overline{m}_y^{\overline{\omega}} y + \overline{m}_{z \, b}^{\overline{\omega}} \right) \right] + \overline{\omega}_y \,_S \,_S [- \mu^2 \Omega - \mu^2 B \Omega] + \right.$ $+ (\alpha_{S \, S} + \alpha_0) \left[\overline{m}_{z \, b}^{\overline{\omega}} \cdot \overline{m}_y^{\overline{\omega}} - \overline{m}_y^{\overline{\omega}} \dfrac{c_y^{\alpha}}{2} - \dfrac{c_y^{\alpha}}{2} \overline{m}_z^{\overline{\omega}} z + \mu^2 \Omega^2 B - \mu \overline{m}_{z \, b}^{\alpha} \right] +$ $+ B \overline{\omega}_z \,_{SS} \left[\mu \left(\dfrac{c_y^{\alpha}}{2} - \overline{m}_z^{\overline{\omega}} z \right) \right] + \mu \left(\mu C \overline{\omega}_z \,_S \,_S - \overline{m}_x^{\overline{\omega}} y \right) \left\{ \beta_{SS} [\mu \Omega \overline{m}_y^{\beta} + \mu B \Omega \overline{m}_{z \, b}^{\alpha}] + \right.$ $+ A \overline{\omega}_y \,_S \,_S \left[\mu B \Omega \left(\dfrac{c_y^{\alpha}}{2} - \dfrac{c_z^{\beta}}{2} \right) \right] + (\alpha_{S \, S} + \alpha_0) \left[- \overline{m}_y^{\beta} \left(\dfrac{c_y^{\alpha}}{2} - \overline{m}_{z \, b}^{\overline{\omega}} \right) \right] -$ $- B \overline{\omega}_z \,_S \,_S \left[\dfrac{c_z^{\beta}}{2} \overline{m}_z^{\overline{\omega}} _b - \dfrac{c_y^{\alpha}}{2} \overline{m}_{z \, b}^{\overline{\omega}} - \dfrac{c_z^{\beta}}{2} \dfrac{c_y^{\alpha}}{2} + \mu^2 \Omega^2 - \mu \overline{m}_{z \, b}^{\alpha} \right] \right\} +$ $+ \mu^2 C \overline{\omega}_y \,_S \,_S \left\{ - \beta_{S \, S} \left[\overline{m}_z^{\alpha} \,_b \left(\overline{m}_y^{\overline{\omega}} + \dfrac{c_z^{\beta}}{2} \right) \right] + A \overline{\omega}_y \,_{SS} \left[- \dfrac{c_y^{\alpha}}{2} \dfrac{c_z^{\beta}}{2} + \dfrac{c_z^{\beta}}{2} \overline{m}_y^{\overline{\omega}} - \right. \right.$ $- \dfrac{c_y^{\alpha}}{2} \overline{m}_y^{\overline{\omega}} - \mu \overline{m}_y^{\beta} + \mu^2 \Omega^2 \right] + (\alpha_{S \, S} + \alpha_0) \left[\overline{m}_{z \, b}^{\alpha} \mu \Omega + \overline{m}_y^{\beta} \cdot A \mu \Omega \right] -$ $- B \overline{\omega}_z \,_S \,_S \left[- A \mu \Omega \left(\dfrac{c_y^{\alpha}}{2} - \dfrac{c_z^{\beta}}{2} \right) \right] \right\}$

TABLE 5 /96

B_0

$$-\overline{m}_x^{\overline{\omega}} A_0 - \overline{m}_x^{\beta} \left\{ -\beta_{\mathrm{SS}} \left[AB\mu^3\Omega^3 + \overline{m}_y^{\overline{\omega}} \overline{m}_z^{\overline{\omega}} {}_{\mathrm{b}} \mu\Omega + \right. \right.$$

$$\left. + \mu^2 B\Omega \overline{m}_z^{\gamma} {}_{\mathrm{b}} \right] + \overline{\omega}_y {}_{\mathrm{S}} {}_{\mathrm{S}} \left[+ \mu^2\Omega \overline{m}_y^{\overline{\omega}} - \mu^2 B\Omega \frac{c_y^{\alpha}}{2} \right] +$$

$$+ (a_{\mathrm{c1}} - a_0) \left[\frac{c_y^{\alpha}}{2} \overline{m}_z^{\overline{\omega}} \overline{m}_y^{\overline{\omega}} + \mu^2\Omega^2 B \frac{c_y^{\alpha}}{2} + \mu \overline{m}_z^{\gamma} {}_{\mathrm{b}} \overline{m}_y^{\overline{\omega}} \right] -$$

$$- B\overline{\omega}_z {}_{\mathrm{S}} {}_{\mathrm{S}} \left[+ \frac{c_y^{\alpha}}{2} \overline{m}_z^{\overline{\omega}} {}_{\mathrm{b}} \mu + \mu^3 A\Omega^2 + \mu^2 \overline{m}_z^{\alpha} {}_{\mathrm{b}} \right] +$$

$$+ \mu \left[\mu C\overline{\omega}_z {}_{\mathrm{S}} {}_{\mathrm{S}} - \overline{m}_x^{\overline{\omega}} y \right] \times$$

$$\times \left\{ -\beta_{\mathrm{SS}} \left[\mu\Omega \overline{m}_y^{\beta} \overline{m}_z^{\overline{\omega}} {}_{\mathrm{b}} + \mu B\Omega \overline{m}_z^{\alpha} {}_{\mathrm{b}} \frac{c_z^{\beta}}{2} \right] + \right.$$

$$+ A\overline{\omega}_y {}_{\mathrm{S}} {}_{\mathrm{S}} \left[\mu^3\Omega^3 B - \mu B\Omega \frac{c_y^{\alpha}}{2} \frac{c_z^{\beta}}{2} + \mu^2\Omega \overline{m}_y^{\beta} \right] +$$

$$+ (a_{\mathrm{S}} {}_{\mathrm{S}} - a_0) \left[\overline{m}_z^{\overline{\omega}} {}_{\mathrm{b}} \frac{c_y^{\alpha}}{2} \overline{m}_y^{\beta} + \mu^2\Omega^2 B \overline{m}_z^{\gamma} {}_{\mathrm{b}} + \mu \overline{m}_z^{\alpha} {}_{\mathrm{b}} \overline{m}_y^{\beta} \right] -$$

$$- B\overline{\omega}_z {}_{\mathrm{S}} {}_{\mathrm{S}} \left[\frac{c_y^{\alpha}}{2} \frac{c_z^{\beta}}{2} \overline{m}_z^{\overline{\omega}} {}_{\mathrm{b}} - \mu^2\Omega^2 \overline{m}_z^{\overline{\omega}} {}_{\mathrm{b}} + \mu \overline{m}_z^{\gamma} {}_{\mathrm{b}} \frac{c_z^{\beta}}{2} \right] \right\} +$$

$$+ \mu^2 C\overline{\omega}_y {}_{\mathrm{S}} {}_{\mathrm{S}} \left\{ -\beta_{\mathrm{SS}} \left[-\overline{m}_z^{\alpha} {}_{\mathrm{b}} \frac{c_z^{\beta}}{2} \overline{m}_y^{\overline{\omega}} + \mu^2\Omega^2 A \overline{m}_y^{\beta} \right. \right.$$

$$+ \mu \overline{m}_y^{\beta} \overline{m}_z^{\gamma} {}_{\mathrm{b}} \right] + A\overline{\omega}_y {}_{\mathrm{S}} {}_{\mathrm{S}} \left[\frac{c_y^{\alpha}}{2} \frac{c_z^{\beta}}{2} \overline{m}_y^{\overline{\omega}} + \mu \overline{m}_y^{\beta} \frac{c_y^{\gamma}}{2} - \right.$$

$$- \mu^2\Omega^2 \overline{m}_y^{\overline{\omega}} \right] + (a_{\mathrm{S}} {}_{\mathrm{S}} - a_0) \left[-\overline{m}_z^{\gamma} {}_{\mathrm{b}} \mu\Omega \overline{m}_y^{\overline{\omega}} + \right.$$

$$+ \overline{m}_y^{\beta} A\mu\Omega \frac{c_y^{\gamma}}{2} \right] - B\overline{\omega}_z {}_{\mathrm{S}} {}_{\mathrm{S}} \left\{ -A\mu^3\Omega^3 - \mu^2\Omega \overline{m}_z^{\alpha} {}_{\mathrm{b}} + \right.$$

$$+ \frac{c_y^{\alpha}}{2} \frac{c_z^{\beta}}{2} A\mu\Omega \right\} \bigg] \bigg\}$$

The expressions for the coefficients A_3, A_2, A_1 and A_0 can be found from Formulas (2.11)-(2.14); ($\Omega \equiv \overline{\omega}_{x0}$).

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may during the maneuver change as a function of the changes in the parameters of motion. As will be obvious from the following, such a variability in $\overline{\omega}_x$ complicates the type of criteria of stability and in certain instances leads to the appearance of new regions of unstable motion.

The conditions of stability for solving the system of Equations (3.42), according to the criteria of the Hurwitz stability, are the satisfaction of the system of inequalities

$$
\left.
\begin{array}{l}
B_4 > 0; \quad B_3 > 0; \quad B_2 > 0; \quad B_1 > 0; \quad B_0 > 0; \\
R_1 = B_4 B_3 - B_2 > 0; \\
R_2 = R_1 (B_2 B_1 - B_3 B_0) - (B_4 B_1 - B_0)^2 > 0.
\end{array}
\right\}
\qquad (3.44)
$$

Not all of the inequalities (3.44) must be verified in analyzing this stability since a part of them are covered by other stronger conditions. For example, in satisfying the conditions $B_4 > 0$, $B_2 > 0$ and $R_1 > 0$, the inequality $B_3 > 0$, etc., must be satisfied first. However if even one of the inequalities is not satisfied this then indicates an instability of motion in connection with which writing all the criteria of stability in certain instances permits simplifying the problem of determining the conditions of loss of stability of motion. The conditions of stability (3.44) written in the form of relationships between the aerodynamic and inertial characteristics of the aircraft are complex and in practice may be used only in making the calculations on computers. One exception is the criterion of aperiodic stability $B_0 > 0$ which determines the stability in many of the cases that are of practical importance and may be obtained in a rather simple form. The condition $B_0 > 0$ is analogous to the condition of the aperiodic stability $A_0 > 0$ which was given in Section 8, and in satisfying the equation $C = \overline{m}_x^\beta = 0$ is converted into it (see Table 5). The inequality $B_0 > 0$ as we know is a necessary condition of the aperiodic stability of motion of an aircraft in the vicinity of the singular point. From the negativity of the free term of the characteristic equation B_0 it follows that the singular point is a saddle type, i.e., the phase trajectories do not enter into the singular point and the stable motion in the vicinity is not realized.

From the system of Equations (3.42) it follows that the expression for B_0 can be obtained if we expand the determinant

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$$
B_0 = \begin{vmatrix}
\dfrac{c_y^2}{2}; & -\mu; & \mu\overline{\omega}_{x0}; & 0; & \mu_{\,\mathrm{s\,s}}^{\beta} \\[2ex]
-\overline{m}_{z\,\mathrm{b}}^{\,2}; & -\overline{m}_z^{\,\overline{\omega}_z}; & 0; & A\mu\overline{\omega}_{x0}; & A\mu\overline{\omega}_{y\,\mathrm{s\,s}} \\[2ex]
-\mu\overline{\omega}_{x0}; & 0; & -\dfrac{c_z^{\beta}}{2}; & -\mu; & \mu(\alpha_0 + \alpha_{\mathrm{s\,s}}) \\[2ex]
0; & -\mu B\overline{\omega}_{x0}; & -\overline{m}_y^{\,\beta}; & -\overline{m}_y^{\,\overline{\omega}_y}; & -\mu B\overline{\omega}_{z\,\mathrm{s\,s}} \\[2ex]
0; & -\mu C\overline{\omega}_{y\,\mathrm{s\,s}}; & -\overline{m}_x^{\,\beta}; & -\left(\mu C\overline{\omega}_{z\,\mathrm{s\,s}} + \overline{m}_x^{\,\overline{\omega}_y}\right); & -\overline{m}_x^{\,\overline{\omega}_x}
\end{vmatrix}. \qquad (3.45)
$$

In determinant (3.45) the minor is set off by the dotted line by the use of which the free term of the characteristic equation A_0 in Section 8 is computed for the motion of an aircraft during a steady rolling turn with $\overline{\omega}_x = \mathrm{const}$. With the direct expansion of the determinant (3.45), rather complex expressions are obtained. For simplification of the computations let us do the following. We convert the right-hand side of the equation of equilibrium of moments relative to the longitudinal axis of the aircraft OX_1 to a single-parameter type, after writing it in the form of a function of one parameter of motion (increasing the angular rolling velocity $\overline{\omega}_x$). After such a transformation, if we set this function into determinant (3.45) instead of the last line we will reduce it to a form which is convenient for computation. Let us make these computations in greater detail.

We look at the equation of motion of an aircraft relative to the longitudinal axis in the variations

$$
\frac{d\overline{\omega}_x}{d\tau} = \overline{m}_x^{\,\overline{\omega}_x}\overline{\omega}_x + \overline{m}_x^{\,\beta}\beta + C\overline{\omega}_{y\,\mathrm{s\,s}}\omega_z + C\overline{\omega}_{z\,\mathrm{s\,s}}\frac{d\omega_y}{}. \qquad (3.46)
$$

Let us divide in Equations (3.46) the terms which correspond to quasistatic changes in the parameters of motion. We can express for this purpose all the parameters of the disturbed motion β, $\overline{\omega}_z$, $\overline{\omega}_y$ which are included in Equation (3.46) through the increase in the value $\overline{\omega}_x$. The increments of the variables which enter into (3.45) can be written in the form of several "quasistatic" changes connected with the change in the value of the variation $\overline{\omega}_x$ and the dynamic changes which depend on the derivatives of the parameters of motion in time. For small variations of $\overline{\omega}_x$ the "quasistatic" changes in the parameters of motion β, $\overline{\omega}_y$, etc., may be written in the following form:

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$$\tilde{\beta} = \left(\frac{\partial \beta_{ss}}{\partial \Omega}\right)_{\bar{\omega}_x = \bar{\omega}_{x0}} \cdot \bar{\omega}_x;$$

$$\tilde{\omega}_y = \left(\frac{\partial \bar{\omega}_{yss}}{\partial \Omega}\right)_{\bar{\omega}_x = \bar{\omega}_{x0}} \cdot \bar{\omega}_x;$$

$$\tilde{\omega}_z = \left(\frac{\partial \bar{\omega}_{zss}}{\partial \Omega}\right)_{\bar{\omega}_x = \bar{\omega}_{x0}} \cdot \bar{\omega}_x.$$

(3.47)

The variables $\tilde{\beta}$, $\tilde{\omega}_y$ and $\tilde{\omega}_z$ would seem to describe changes in the respective parameters of motion with infinitely slow change in the angular rolling velocity by a value of $\bar{\omega}_x$. The total changes in the variables β, $\bar{\omega}_y$, $\bar{\omega}_z$ may be written in the following form:

$$\beta = \tilde{\beta} + g_\beta \left(\bar{\omega}_x, \dot{\alpha}, \ldots\right);$$

$$\bar{\omega}_y = \tilde{\omega}_y + g_{\bar{\omega}_y}\left(\bar{\omega}_x, \dot{\alpha}, \ldots\right);$$

$$\bar{\omega}_z = \tilde{\omega}_z + g_{\bar{\omega}_z}\left(\bar{\omega}_x, \dot{\alpha}, \ldots\right),$$

(3.48)

where the functions g_β, $g_{\bar{\omega}_y}$, $g_{\bar{\omega}_z}$ describe the "dynamic" components of change of these variables and vanish in steady motion. If we substitute Expressions (3.47) and (3.48) into Equation (3.46) and group the terms by $\bar{\omega}_x$, we find the equation for the change in the variation $\bar{\omega}_x$ in the form

$$\frac{d\bar{\omega}_x}{d\tau} = \left[\bar{m}_x^{\omega_x} + \bar{m}_x^\beta \left(\frac{\partial \beta_{ss}}{\partial \bar{\omega}_x}\right)_{\bar{\omega}_x = \bar{\omega}_{x0}} + \right.$$
$$\left. + C\left(\frac{\partial \bar{\omega}_{yss} \cdot \bar{\omega}_{zss}}{\partial \bar{\omega}_x}\right)_{\bar{\omega}_x = \bar{\omega}_{x0}}\right]\bar{\omega}_x + f\left(\bar{\omega}_x; \dot{\alpha}, \ldots\right),$$

where $f(\bar{\omega}_x,\ldots)$, analogous to g_i, describes the dynamic components of the function. Direct verification is easy to obtain that the expression in the brackets is equal to the derivative $(\partial \Delta \bar{m}_x / \partial \omega_x)\omega_x$ = ω_{x0} with the reverse sign. From the transformations it follows that the free term of the characteristic equation of the systems of Equation (3.42) [determinant (3.45)] can be written in the form

$$B_0 = \begin{vmatrix} A_0 & & N \\ & & \\ 0 & & \left(\frac{\partial \Delta \bar{m}_x}{\partial \bar{\omega}_x}\right)_{\omega_{x0}} \end{vmatrix},$$

(3.49)

where A_0 is the determinant for the free term of the system of
Equations (1.33) which were studied in detail in analyzing the
stability of steady rotation of an aircraft at a constant angular
rolling velocity. If we expand the determinant (3.49) from the
elements of the last line we find the final expression for the
free term of the characteristic equation B_0 in the form

$$\dot{B}_0 = A_0 \left(\frac{\partial \Delta \overline{m}_x}{\partial \overline{\omega}_x} \right)_{\overline{\omega}_{x0}} . \tag{3.50}$$

On the basis of Equation (3.50) the necessary condition of the
stability $B_0 \geq 0$ may be written in greater detail in the following
form which is convenient for use. The motion is aperiodically
stable if the following inequalities are satisfied:

$$\frac{\partial \Delta \overline{m}_x}{\partial \overline{\omega}_x} \geqslant 0 \quad \text{when} \quad A_0 > 0; \tag{3.51}$$

$$\frac{\partial \Delta \overline{m}_x}{\partial \overline{\omega}_x} \leqslant 0 \quad \text{when} \quad A_0 < 0. \tag{3.52}$$

Since the criterion is quite important, let us look briefly at
the second method of obtaining it. We can determine from the system
of equations in variations (3.42) the increase in the value of $\Delta \overline{\omega}_x$
corresponding to the small increment of the external rolling moment
($\overline{m}_x$), which we can add to the right-hand side of the last equation
in system (3.42). Let us set the derivatives in Equations (3.42)
equal to zero and seek the solution for $\Delta \overline{\omega}_x$ from the general rules
in the form of a ratio of determinant (2.45) (in which the extreme
right-hand column is substituted by the column of zeros in all lines
other than the last one where the value $\Delta \overline{m}_x$ is found) to the free
term of the characteristic equation B_0:

$$\Delta \overline{\omega}_x = \frac{1}{B_0} \begin{vmatrix} A_0 & 0 \\ 0 & \Delta m_x \end{vmatrix} . \tag{3.53}$$

If we expand the determinant in Expression (3.53) we obtain

$$\Delta \overline{\omega}_x = \frac{A_0 \Delta \overline{m}_x}{B_0} \quad \text{or} \quad B_0 = A_0 \left(\frac{\Delta \overline{m}_x}{\Delta \overline{\omega}_x} \right) .$$

Hence, if we proceed to the limit and direct $\Delta \overline{\omega}_x$ and $\Delta \overline{m}_x$ toward
zero we find the same expression for B_0, just as previously, in the
form

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$$B_0 = A_0 \lim_{\Delta \overline{\omega}_x \to 0} \left(\frac{\Delta \overline{m}_x}{\Delta \overline{\omega}_x} \right) = A_0 \cdot \left(\frac{\partial \overline{m}_x}{\partial \overline{\omega}_x} \right)_{\overline{\omega}_x 0}.$$

It should be noted that the obtained criteria of stability are precise and not approximate and may be verified by direct computation.

It is easy to see that inequalities (3.51) and (3.52) are a generalization of the condition of aperiodic stability of motion of an aircraft obtained in Section 8 during the steady rotation relative to the longitudinal axis. In fact, in order that the motion of the aircraft at a constant angular rolling velocity be stable, it is necessary that in the equation of equilibrium of moments relative to the longitudinal axis,

$$\frac{d \overline{\omega}_x}{d \tau} = \overline{m}_x^{\overline{\omega}_x} \overline{\omega}_x + \Delta \overline{m}_x.$$

(3.54)

that the value of $\overline{m}_x^{\omega x}$ be negative.

From Equation (3.54) we find the expression for the derivative $\partial \Delta \overline{m}_x / \partial \overline{\omega}_x$:

$$\frac{\partial \Delta \overline{m}_x}{\partial \overline{\omega}_x} = - \overline{m}_x^{\overline{\omega}_x}.$$

(3.55)

From Expression (3.55) it follows that the derivative does not depend on the size of the angular rolling velocity and is always positive. In this case from Expressions (3.51) and (3.52) we find the condition of stability in the form

$$A_0 \geqslant 0,$$

which agrees with the criterion of aperiodic instability given in Section 8.

The inequality $B_0 \geq 0$ and conditions (3.51) and (3.52) are necessary but not sufficient criteria of stability. This means that for the stability of motion in the vicinity of the singular point their satisfaction is necessary; however, it is insufficient for proof of the stability of motion. From the inequality $B_0 \geq 0$ it follows only that in the characteristic equation there is no odd number of real positive roots. However, we can not exclude the possibility of the presence of an even number of positive real roots or any number of complex conjugate roots with a positive real part. Usually the characteristic Equation (3.43) has no more than one positive root with respect to which the criteria of stability (3.51)

and (3.52) permit determining the aperiodic stability of motion of an aircraft in the vicinity of the singular point, i.e., the isolation of saddle-type singular points.

The criterion of aperiodic instability given permits finding a rather important type of singular point quite easily, i.e., the saddle-type singular points which in a large number of instances divide the regions of attraction of singular points of "focus" type. The other criteria of stability (3.44) cannot be simplified.

Let us look briefly at several properties of the coefficients and roots of the characteristic Equation (3.43). Let us look at the behavior of the free term·of the characteristic equation in the /102 vicinity of the critical angular rolling velocities. In analyzing the motion of an aircraft with $\bar{\omega}_x$ = const we found that the free term of the characteristic equation vanishes and then changes sign with transition through the critical angular rolling velocities. In the general case $m_x^\beta \neq 0$ the changes in B_0 in the vicinity of the critical velocities may be studied in the following way. It is easy to show that the function $\Delta \bar{m}_x(\bar{\omega}_x)$ can be written in the form

$$\Delta \bar{m}_x = \frac{G(\bar{\omega}_x)}{A_0(\bar{\omega}_x)},$$

(3.56)

where the function $G(\bar{\omega}_x)$ has no zeros corresponding with the zeros $A_0(\bar{\omega}_x)$.

By differentiating the function $\Delta \bar{m}_x$ over $\bar{\omega}_x$, we find

$$\frac{\partial \bar{m}_x}{\partial \bar{\omega}_x} = -\frac{G \frac{\partial A_0}{\partial \bar{\omega}_x}}{A_0^2} + \frac{\frac{\partial G}{\partial \bar{\omega}_x}}{A_0}.$$

(3.57)

Let us substitute Expression (3.57) into the formula for B_0:

$$B_0 = -\frac{G(\bar{\omega}_x)}{A_0} \cdot \frac{\partial A_0}{\partial \bar{\omega}_x} + \frac{\partial G(\bar{\omega}_x)}{\partial \bar{\omega}_x}.$$

(3.58)

With transition through the critical rolling velocity the value A_0 vanishes and the derivative $\partial A_0/\partial \bar{\omega}_x$ is non-zero. Hence it follows that at critical angular rolling velocities, B_0 has a terminal of first order, i.e., in the approach of the parameter $\bar{\omega}_x$ to the critical values, the roots of the characteristic equation begin to grow without limit and with passage through the critical angular velocity at least one root undergoes disruption and changes sign.

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From Table 5 it is obvious that the coefficient B_4 of the characteristic Equation (3.43) does not depend on the sign of the angular rolling velocity, $\bar{\omega}_x$. On the other hand, it is known that the sum of the real parts of all roots is equal to this coefficient, i.e.,

$$B_4 = \sum_i (2\xi_i + \lambda_i) = \text{const.}$$

(3.59)

Thus, from Equation (3.59) it follows that if the real part of even one of the roots of the characteristic equation begins to grow, in particular if it tends to an infinitely large value then there is always a root whose real part tends to infinitely large values of the opposite sign.

15. Investigation of the Motion of an Aircraft in Phase Space. Types of Rolling Maneuvers with the Simultaneous Control by Ailerons and Elevator

Let us proceed to an investigation of the possible types of controlled spatial motion of an aircraft. It is possible to obtain the most general representations of the spatial motion of an aircraft by using the methods of the qualitative theory of differential equations. However, in such case we must introduce limitations to the possible types of control deflections. In fact, in order that the motion be analyzed in phase space it is essential that the system of equations of motion be autonomous, i.e., that they do not contain variables of time parameters. One exception involves intermittent - gradual changes in which case at the moment of a sudden change in parameters the phase picture of the motion is also changed respectively. With respect to the comments below we will look at the maneuvers of an aircraft which are carried out with the simultaneous gradual deflection of controls in various combinations. This somewhat limits the region of applicability of the results obtained since the limitation in speed of deflections of the controls is not taken into account. However, the solutions thus found permit obtaining a qualitative concept as to the dynamic characteristics of an aircraft in the general case.

Both the transient conditions and the steady values of the parameters of motion of an aircraft are determined by the values of the control deflection. In such case, each combination of the gradual control deflection (δ_a, δ_e, δ_r) is represented by a given system of singular points in phase space of the parameters of motion (α, β, $\bar{\omega}_x$, $\bar{\omega}_y$, $\bar{\omega}_z$) and the values of these parameters (α_0, β_0, $\bar{\omega}_{x0}$, $\bar{\omega}_{y0}$, $\bar{\omega}_{z0}$) at the moment of time preceding deflection of the controls are the initial ones. It is obvious that the motion of an aircraft with the simultaneous gradual deflection of the controls can be studied in phase space by taking into account that with each new deflection of the controls, the phase picture and the position of the singular points vary. In this case, the initial position of the figurative point in phase space depends on the

interval of time between the moments of the control deflections.

In the present section we analyze the dependence of the number and type of singular points on the values of the aileron and elevator deflections and briefly look at the motion in the small vicinity of each singular point. Control of the aircraft using ailerons and rudder will be looked at separately. With respect to this, in the analysis we can limit ourselves to three basic parameters (δ_a, α_0 and $\Delta\alpha_b$), which characterize the control of an aircraft on which the interrelationship of the longitudinal and lateral motions depend with the simultaneous control of pitching and rolling.

The motion of an aircraft in the vicinity of the singular point is described by the linear differential equations (3.42) given in Section 14. The basic method of analyzing the qualitative picture of the motion will be the determination of the type of singular point and finding the dependence of the roots of the characteristic equation of the system of Equations (3.42) on the value of the angular rolling velocity at all singular points.

In Section 11 we introduced a definition of the singular points on which are based the "spatial focus" and the "spatial saddle". These types of singular points correspond to the two characteristic types of motion. If the singular point is a focus, the phase trajectory either "is wound" on it, or is "unwound" as a function of whether all the real parts of the roots of the characteristic equation are negative or if there are positive real parts of the complex-conjugate roots. In the case when the singular point is a spatial saddle, the phase trajectory approaches it at a certain minimal distance after which it remains at that point. One exception consists of the phase trajectories which lie on the separatrix surfaces that pass through the singular points of "saddle" type. It is noted in Section 11 that the separatrix surfaces defined in the phase space regions with different types of phase pictures in particular define the regions of attraction of the singular point of "focus" type. With respect to this the determination of the singular points of the "spatial-saddle" type is of special interest. The criterion which permits defining saddle-type singular points was given in Section 14 and includes the fact that the singular point is a saddle if the following inequality is satisfied:

$$\left.\begin{array}{ll} \dfrac{\partial \Delta \bar{m}_x}{\partial \bar{\omega}_x} > 0, & \text{if} \quad A_0 < 0; \\[2ex] \dfrac{\partial \Delta \bar{m}_x}{\partial \bar{\omega}_x} < 0, & \text{if} \quad A_0 > 0. \end{array}\right\} \qquad (3.60)$$

Before we proceed to the general case of motion of an aircraft, let us look at the specific case when $\bar{m}_x^\beta = C = 0$. In this case

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the criterion for the existence of a saddle-type singular point (3.60) is simplified (see Section 14) to the form $A_0 < 0$.

On Figure 3.16 is constructed the function $\Delta\bar{m}_x(\bar{\omega}_x)$ for this case and the corresponding dependence of the root of the characteristic equation on the angular rolling velocity $\bar{\omega}_x$. From Figure 3.16 it follows that the relationship between $\Delta\bar{m}_x$ and the parameters of motion of an aircraft is a unique one and does not depend on the pitching motion of the aircraft (α_0, $\Delta\alpha_b$). In this case the motion outside the range of the critical rolling velocities is stable and in the range between the critical rolling velocities there exist saddle-type singular points and the motion for the entire parameters other than $\bar{\omega}_x$ is unstable.

The dynamics of an aircraft in the general case are described /105 much more complexly when $\bar{m}_x^\beta \neq 0$. In this case the type of division in phase space of the singular points depends on the parameters of the longitudinal motion which is associated particularly with the increasing complexity of the type of function $\Delta\bar{m}_x(\bar{\omega}_x)$ that was analyzed in Section 13.
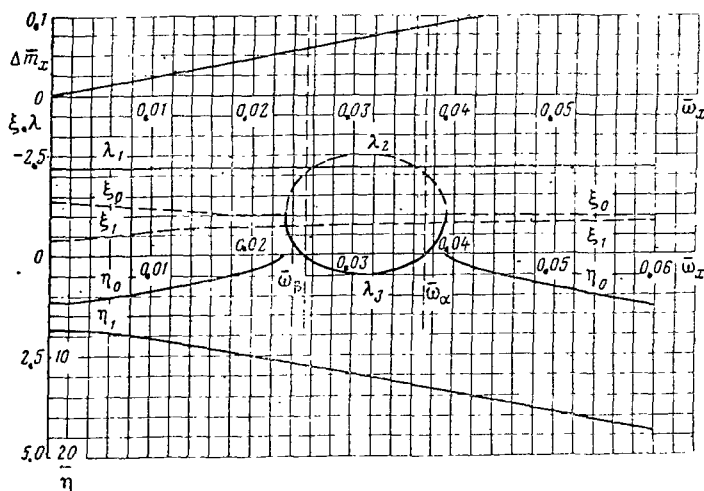
Fig. 3.16.

Below we shall look at the different types of spatial maneuvers of an aircraft. As the type of spatial maneuver we mean the possible types of phase pictures which correspond to a fixed combination of deflections of the aircraft controls for pitching (in the general case for pitching and yawing) for all possible values of deflections of the ailerons. In accordance with this the types of spatial maneuvers are determined by the pitching control of the aircraft and the specific realization of the motion even now depends on the size of the aileron deflection. In Section 13 we looked at the first part of this problem, i.e., we found the basic types of the dependence of the required values of the aileron deflections on the parameters of control of the aircraft for pitching, i.e., the

types of the function $\Delta \bar{m}_x(\bar{\omega}_x, \alpha_0, \Delta\alpha_b)$.

The function $\Delta \bar{m}_x(\bar{\omega}_x, \alpha_0, \Delta\alpha_b)$ to a large degree determines the type of motion of the aircraft, particularly the condition of its aperiodic stability (see Section 14). In this respect for the basis of investigation of the phase pictures in the space of the parameters of motion of an aircraft, we take the corresponding function $\Delta m_x(\bar{\omega}_x, \alpha_0, \Delta\alpha_b)$ and for each type of such function (type of maneuver) we find the dependence of the number and type of singular points on the size of the deflection of the ailerons and analyze the condition of the possibility of motion in the vicinity of the obtained singular points.

Type B Rolling
Maneuver

Definitions

● Stable spatial
  focal point.

o Unstable spatial
  focal point.

¤ Spatial saddle
  point.
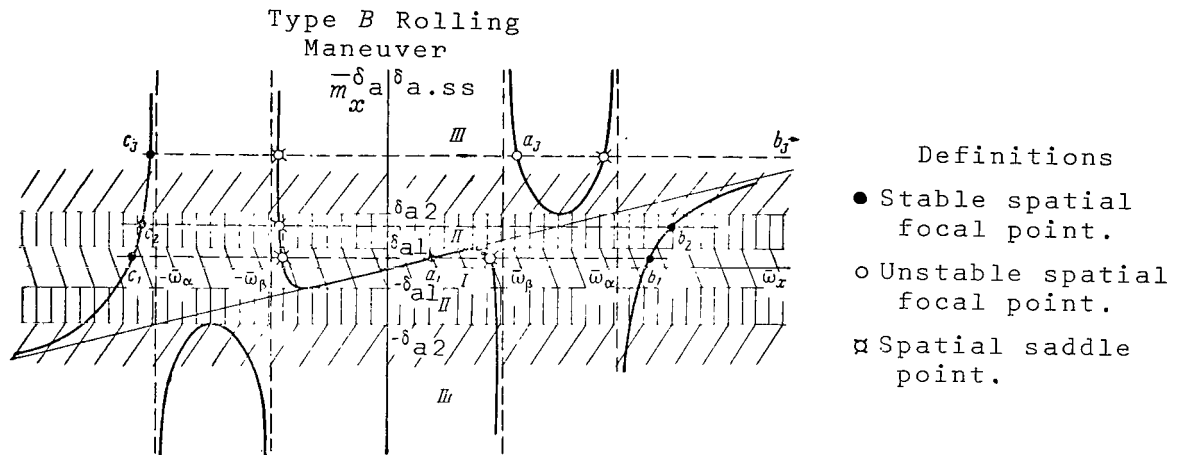
Fig. 3.17

Let us note several general rules for location of the singular points in phase space. We can show that if $J_y = J_z (C = 0)$, then the singular points of focus type alternate with the change in the parameter ω_x with saddle-type singular points. In fact, in this case the character of the disruption of the function $\Delta m_x(\bar{\omega}_x)$ is such that the derivative $\partial m_x/\partial \omega_x$, in passing through the critical angular rolling velocities, does not change sign but the value $A_0(\omega_x)$ does. On the other hand, in the regions of continuous change of the function $\Delta m_x(\bar{\omega}_x)$ it is obvious that if $\Delta m_x(\bar{\omega}_x)$ assumes identical values for two different sizes of the angular rolling velocity $\bar{\omega}_x$ between the angular velocities due to the continuousness of the function the derivative changes sign. Hence, by taking into account criterion (3.51) we find that the singular points of focus and saddle-type do in fact alternate for the parameter $\bar{\omega}_x$. This property in the general case in the vicinities of the critical angular velocities is disrupted. In fact when $C \neq 0$, with passage through the critical angular velocity, $\Delta m_x(\bar{\omega}_x)$ tends to infinity but does not change sign thus indicating the simultaneous change in the sign of the derivative $\partial m_x/\partial \omega_x$ and $A (\bar{\omega}_x)$. According to

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criterion (3.60) in this case the type of the singular point, both with smaller than critical and larger than critical angular velocities, is not changed. Let us note that all the remaining parameters of motion, with passage through the critical rolling velocity, change sign, i.e., such singular points seem to be arranged "antisymmetrically" in phase space relative to the axis $O\bar{\omega}_x$. When the coefficient C tends toward zero, these pairs of singular points in the vicinities of the critical angular rolling velocities for all of the parameters of motion other than $\bar{\omega}_x$ remain a distance apart from one another and at the limit go to infinity.

Let us proceed to an anlysis of the properties of the possible types of spatial maneuvers of an aircraft on an example of types B, C, D and E which are of the most practical interest (see Section 13).

Since the motion of an aircraft in the immediate vicinity of the critical rolling velocities in the general case is of basic theoretical interest then in all computations we assume that $C = 0$. As noted in Section 13, such an assumption leads to errors only in analyzing motion in quite small vicinities of the critical rolling velocities. However, the number of possible types of maneuvers which are necessary for analysis are considerably decreased.

Type B Rolling Maneuvers

The function $\Delta \bar{m}_x(\bar{\omega}_x)$ for the entire range of changes in the angular rolling velocity

$$-\infty < \bar{\omega}_x < \infty \tag{3.61}$$

is shown on Figure 3.17. Graphs of the regions of values of the parameters of motion in the longitudinal plane (α_0, $\Delta\alpha_b$), for which the function $\Delta \bar{m}_x(\bar{\omega}_x)$ has such a shape, are plotted on Figure 3.18. For this type of rolling maneuver the initial one is flight when the major inertial axis of the aircraft is located under the velocity vector. In this case, as a function of the value of the aileron deflection, the motion of the aircraft will be different.

From Figure 3.17 it follows that each value of the angular rolling velocity $\bar{\omega}_x$ is represented uniquely by a certain value of the aileron deflection (the quantity $\Delta \bar{m}_x$). With respect to this, as an example of the basic parameter, we can look at the value of the angular rolling velocity $\bar{\omega}_x$.

On Figure 3.19 are given graphs of the trajectories of the roots of the characteristic Equation (3.43) plotted for the case of motion under analysis when $\bar{\omega}_x > 0$. The trajectories of the roots for $\bar{\omega}_x < 0$ are obtained as a mirror reflection of the semiplane $\bar{\omega}_x > 0$.

From the very method of finding the roots, using the character-
istic Equation (3.43), it follows that they describe the dynamic
properties of motion in the immediate vicinity of the respective
singular point that is determined by the angular rolling velocity
ω_x. With the help of Figure 3.19 for a given deflection of the
ailerons we can find the values of ω_x for which the motion of an
aircraft is possible and consequently all singular points in the
values of the roots of the characteristic equation on which depend
the motion in the immediate vicinity of this singular point where
the linearized Equations (3.42) are valid. For a type B maneuver,
we can define three basic regions
of values of the lateral moment
($\Delta \overline{m}_x$), located symmetrically rela-
tive to the axis $\Delta \overline{m}_x = 0$, which
differs in the number and type of
singular points and corresponds to
the different character of the
motion of an aircraft with control
by the ailerons. Let us look at
these regions in more detail.

The conventional definitions
of the singular points are given
in Figure 3.17.

First region
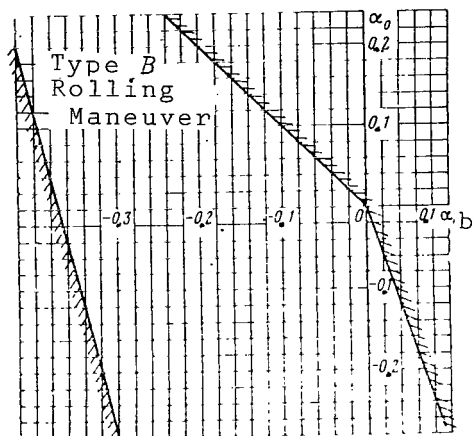
Fig. 3.18

$$|\Delta \overline{m}_x| \leqslant |\Delta \overline{m}_{x1}|.$$ (3.62)

In this range of aileron deflections (the values of the con-
trolling moment, acting relative to the longitudinal axis) there
are five singular points of which three are singular points of
stable-focus type and two of saddle-type and three regions of stable-
focus type and two of saddle-type and three regions of stable motion
relative to the singular points a, b, and c which are defined by
saddle-type singular points (see Fig. 3.17). What kind of aircraft
motion for each value of the aileron deflection ($\Delta \overline{m}_x$) from the in-
terval (3.62) will in fact be realized depends on the initial con-
ditions during the maneuver. In this case the values of the parame-
ters of motion, particularly the value of the angular rolling veloc-
ity ω_x in the steady system will be different (see Fig. 3.17). For
example, in carrying out a rolling maneuver from the conditions of
flight with zero initial conditions based on the basic parameters
of motion, the point of "attraction" for the solution is the singu-
lar point a_1. The character of the dynamic properties of motion in
the vicinity of this singular point may be estimated according to
the values of the roots of the characteristic equation given on
Figure 3.19. For other initial conditions the points of "attraction"
of the solution may be the singular points c_1 and b_1. In practice,
motion in the vicinity of all three stable singular points is real-
ized. As noted above, motion in the vicinity of the singular point
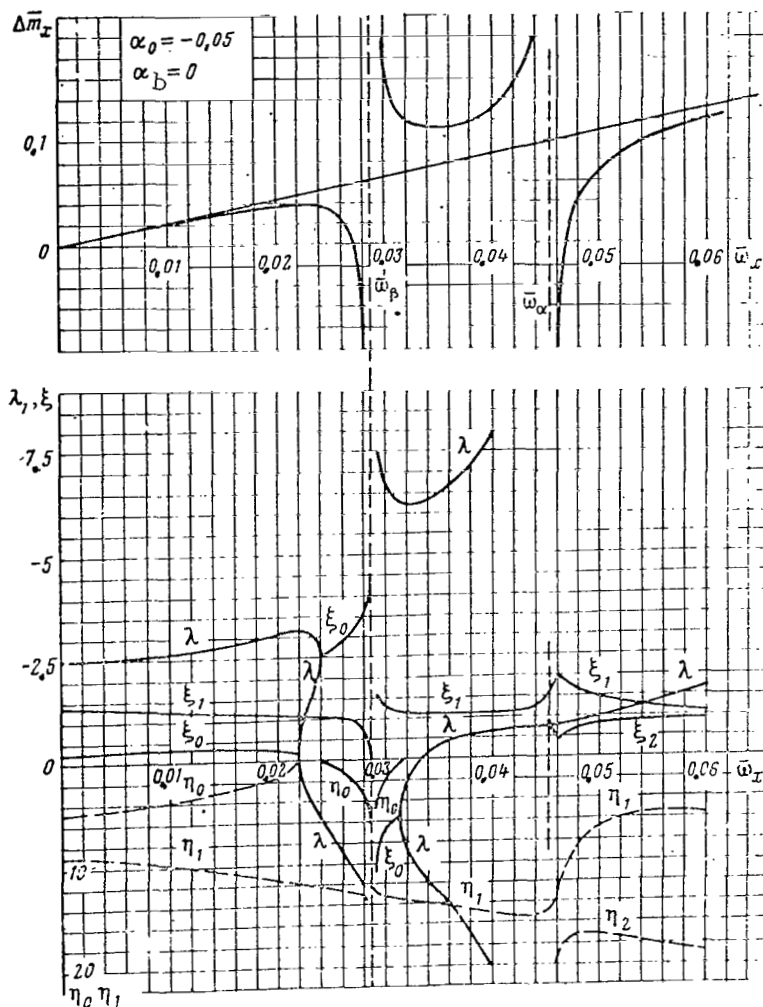
Fig. 3.19
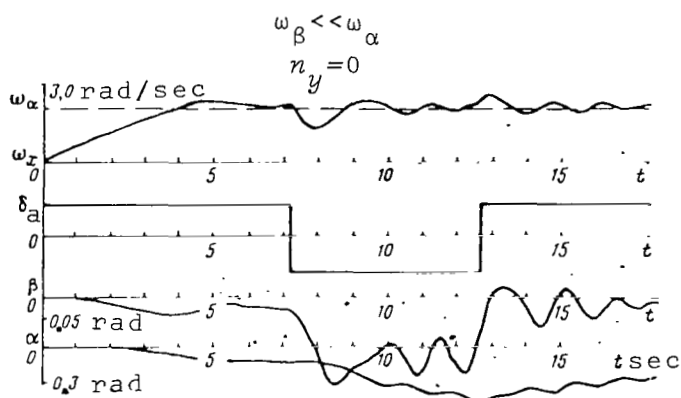
Fig. 3.20

b_1 is realized in that case when the gradual deflection of the ailerons is accomplished under zero initial conditions based on the angular rolling velocity $\bar{\omega}_x$ and the remaining parameters of motion. In carrying out a rolling maneuver by deflection of the ailerons by a value which lies in region II, the motion of the aircraft is accomplished in the vicinity of the singular point b_2. Change under these conditions of the value $\Delta \bar{m}_x$ up to values from region I creates conditions for realizing motion in the vicinity of the singular point b_1. Under analogous conditions motion of an aircraft is realized in the vicinity of the singular point c_1; only for this is the initial application to the aircraft of a negative control moment of roll ($\Delta \bar{m}_x$) necessary from region II. From Figure 3.17, it is clear that motion in the vicinity of the singular points of type b_1 and c_1 can be realized also with the ailerons placed in the neutral position. Such conditions of motion of an aircraft, when it practically loses controllability by the ailerons, have been called systems of "inertial rotation" or a system "autorotation" of the aircraft. This latter designation is especially widely used in the foreign literature.

Figure 3.20 shows the graph of the change in the basic parameters of motion of an aircraft when its motion (after change of sign of the deflection of the ailerons) is realized relative to the singular point of b_1 type (see Fig. 3.17). Such systems are studied in greater detail in analyzing the dynamics of controlled motion of an aircraft in Chapter IV.

Let us note that the systems of "inertial rotation" of an aircraft corresponding to the motion in the vicinity of the singular points on the static curve b and c (see Fig. 3.17) were observed in flight. From the point of view of aircraft dynamics the motion in the vicinity of the singular points of c_1 type is interesting in that the positive value $\Delta \bar{m}_x$ is represented by a negative angular rolling velocity, i.e., a rolling velocity which is opposite to the usual sign.

Second region

$$|\Delta \bar{m}_{x1}| \ll |\Delta \bar{m}_x| \ll |\Delta \bar{m}_{x2}|.$$

(3.63)

In the range of aileron deflections (3.63) there are three singular points, two of which (b_2, c_2) (see Fig. 3.17) are stable focal points and correspond to the stable motion of an aircraft, and one is a saddle-type singular point. The separatrix surfaces which pass through the singular point divide the phase space into two regions, in each of which the motion is stable. With zero initial conditions and gradual deflection of the ailerons the point of "attraction" is the singular point b_2.

An example of the transient condition in the vicinity of the singular point b_2 is the beginning of motion shown on Figure 3.20

(prior to change of the ailerons).

Region II (see Fig. 3.17) creates a "window" in the static curves and permits during control "realizing" motion in the vicinity of the singular points b_1, b_2, b_3 on curve b and in the vicinity of the singular points c_1, c_2, c_3 on curve c.

Third region

$$|\Delta \overline{m}_x| \geqslant |\Delta \overline{m}_{x2}|. \qquad (3.64)$$

In this region there are five singular points since two singular points again appear additionally. In this case the singular points c_3 and b_3 are stable focal points, and the singular point a_3 is an unstable focal point. The regions of attraction of the focal points are separated by saddle-type singular points. With zero initial conditions and gradual deflection of the ailerons by a value which satisfies condtion (3.64) at the beginning of the transient condition motion is realized relative to the unstable focal point. However, since in the vicinity of the singular point a_3 there are no other singular points other than the saddle-type then as a function of the parameters of aircraft either a certain limiting cycle is established and the motion has the character of undamped nonlinear oscillations or the phase trajectory penetrates the region of attraction of the singular point b_3.

Motion in the vicinity of the singular point b_3 can also be realized by the subsequent deflection of the ailerons from the beginning by a value which satisfies conditions (3.63) and then additionally up to a value which satisfies condition (3.64). In this case the motion is stabilized after the first deflection in the vicinity of the singular point of b_2 type (see Fig. 3.17), and after additional deflection of the ailerons near the singular point of b_3 type.

Motion in the vicinity of the singular point c_3 is realized with the subsequent deflection of ailerons first from the negative part of region II [Eq. (3.63)] and then with change to positive region III [Eq. (3.64)]. In general motion in the vicinity of the singular points c_3 and b_3 can be realized because of the presence of a "window" in region II.

Such are the basic properties of the qualitative picture of possible types of motion of an aircraft during rolling maneuvers of B type. A brief analysis shows that substantial difference in the results following from the total equations of motion in their simplified linear equations of motion. Using the linear equations of motion we were able to determine only the singular points which lie on the asymptote

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$$\delta_a = - \frac{\bar{m}_x^{\bar{\omega}_x} \cdot \bar{\omega}_x}{\bar{m}_x^{\delta_a}}$$

The difference in solutions is clearly obvious in Figure 3.17. Let us note that also with small values of the deflections of ailerons, particularly when $\delta_a = 0$, from linear theory we may find only one singular point, however as was shown above if the initial angular rolling velocity $\bar{\omega}_x$ of the aircraft is high then motion is also possible with undeflected ailerons in the vicinity of the singular points c_1 or b_1. With small values of the deviations of the parameters of motion from zero and small deflections of the controls, linear theory qualitatively describes the motion of an aircraft properly, since in this case the point of "attraction" is the singular point a_1 and the region of its "attraction" is bordered by two saddle-type singular points. However it should be noted that the region of changes in parameters, for which linear approximation gives us sufficient accuracy, depends on the parameters of control of the aircraft for pitching (α_0, $\Delta\alpha_b$).
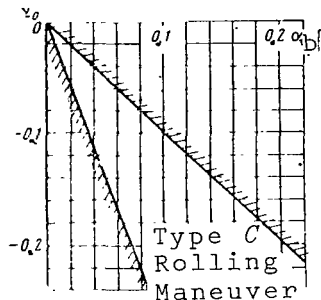
Fig. 3.21

Type C Rolling Maneuvers

Such a maneuver exists if the parameters which characterize the longitudinal controlled motion of an aircraft α_0 and $\Delta\alpha_b$ lie in the region that is shown in Figure 3.21. Such a type of maneuver exists when the rolling turn is carried out from the conditions of flight of the aircraft with a negative G-force ($\alpha_0 < 0$) with a preliminarily small deflection of the control stick ($\Delta\alpha_b > 0$). From the viewpoint of practical flying such a type of maneuver is mainly of theoretical interest. Static curves of the function $\Delta\bar{m}_x(\bar{\omega}_x)$ for a rolling maneuver of C type are shown on Figure 3.22, and the trajectories of the roots on Figure 3.23. From these figures it follows that in the entire region of aileron deflections in phase space of the parameters of motion there are three basic points of /113 stable and unstable spatial focal types and two saddle-type singular points, which define the region of "influence" of these singular points. After deflection of the ailerons for zero initial conditions the figurative point in phase space moves to a point which is an unstable focus. Motion in the vicinity of the singular point a_1 (see Fig. 3.22) is a nonlinear undamped oscillational process, and in phase space there may be a certain limiting cycle that is valid. Motion in the vicinity of the singular points b_1 and c_1 may be obtained by subsequently carrying out a type B maneuver with escape to the angular rolling velocities exceeding $\bar{\omega}_{x2 \text{ crit}}$ and then a type C maneuver, i.e., by the subsequent deflection of the elevator

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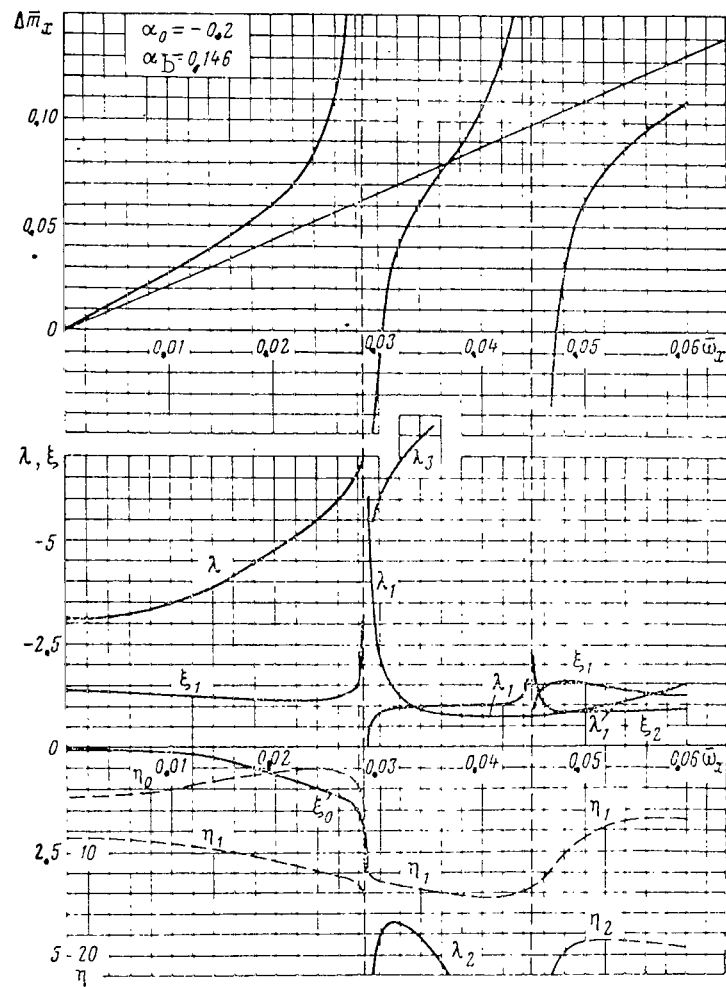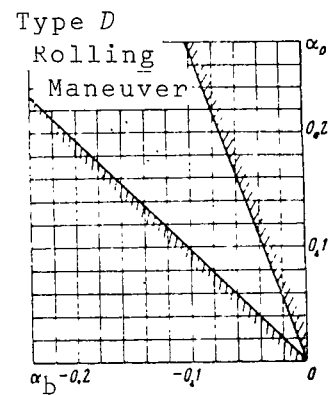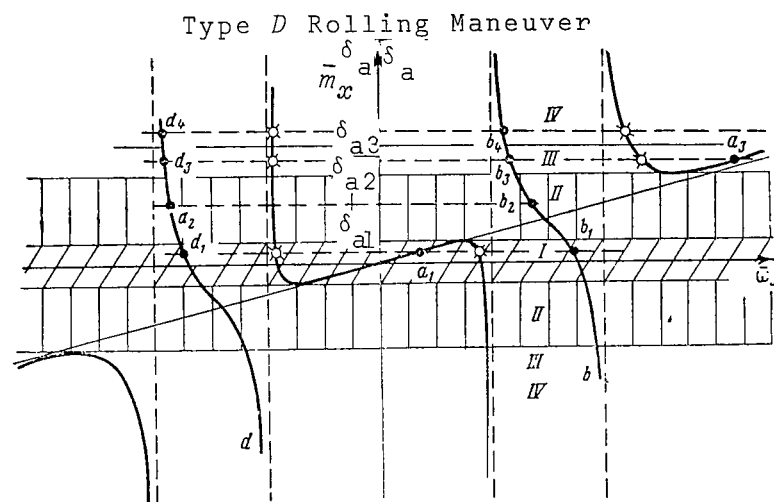
Fig. 3.23

Fig. 3.24

Fig. 3.25

and ailerons.

Type C Rolling Maneuver
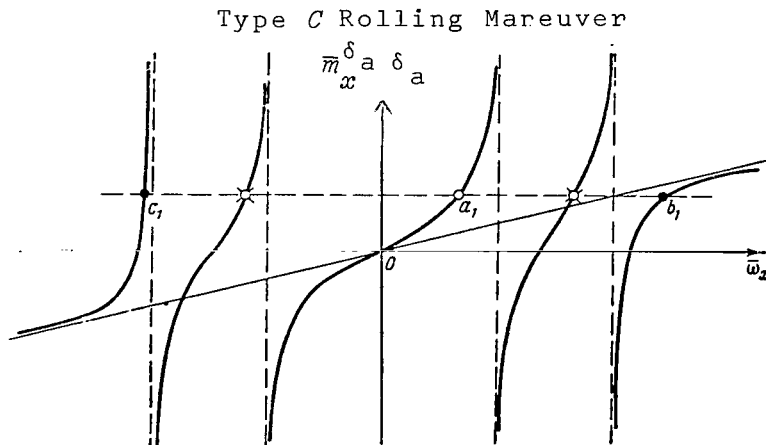
Fig. 3.22

Type D Rolling Maneuver

Type D rolling maneuvers occur if the parameters (α_0, $\Delta\alpha_b$), which characterize the longitudinal control of an aircraft, lie in the region noted on Figure 3.24. In order that the rolling maneuver belong to a D type, control of the aircraft for pitching must be accomplished by deflection of the control stick away ($\Delta\alpha_b < 0$) under conditions of horizontal flight ($\alpha_0 > 0$). The maneuver is mainly of theoretical interest since it is accomplished for a narrow range of parameters (α_0, $\Delta\alpha_b$). The curve showing the function $\Delta\overline{m}_x(\omega_x)$ for type B maneuvers is shown on Figure 3.25 and the trajectories of the roots on Figure 3.26.

From Figure 3.25 it follows that there are four basic ranges of values of $\Delta\overline{m}_x$ which differ in the number and type of singular points.

First region

$$|\Delta\overline{m}_x| \leqslant |\Delta\overline{m}_{r1}|.$$

(3.65)/116

In this region there are five singular points: three singular points of stable focus type, which are separated by two saddle-type singular points. For zero initial conditions the motion is accomplished relative to the singular point corresponding to the stable focus; as a result of the system there are observed damped oscillations. Stable solutions in the vicinity of the singular points b_1 and d_1 (see Fig. 3.25) may be obtained because of the presence of a "window" in the second region which is examined below with the respective changes in the ailerons analogous to that which was done

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in analyzing the type B maneuver.

Second region

$$|\Delta \overline{m}_{x1}| \leqslant |\Delta \overline{m}_{x}| \leqslant |\Delta \overline{m}_{x2}|. \tag{3.66}/\underline{117}$$

The second region is the "window" which permits penetrating to the singular points on curves b and d. In this region there are two singular points of stable focus type, that are separated by a saddle-type singular point (see Fig. 3.25).

Deflection of the ailerons for zero initial conditions leads to motion in the vicinity of the singular point b_2. The return of the ailerons to neutral position after establishing motion in the vicinity of the singular point b_2 leads to motion in the vicinity of the singular point b_1.

Third region

$$|\Delta \overline{m}_{x2}| \leqslant |\Delta \overline{m}_{x}| \leqslant |\Delta \overline{m}_{x3}|. \tag{3.67}$$

In the third region there are five singular points; three points of stable focus type and two saddle-type singular points, which describe the focal point. Under zero initial conditions, motion occurs relative to the stable focus b_3.

Fourth region

$$|\Delta \overline{m}_{x}| > |\Delta \overline{m}_{x3}|. \tag{3.68}$$

In the fourth region the number of singular points is the same as in the third region but the singular point of stable focus type on the branch of the static function b (see Fig. 3.25) becomes an unstable focal point.

Type E Rolling Maneuvers

On Figure 3.27 the range of values $(\alpha_0, \Delta \alpha_b)$ is plotted for which type E rolling maneuvers are realized. Maneuvers of this type are characteristic of the controlled rolling motion of an aircraft that is carried out from conditions of horizontal flight or flight with a positive normal G-force. Type F rolling maneuvers for the neighboring region of parameters $(\alpha_0, \Delta \alpha_b)$ are little different from type E maneuvers and therefore are not studied separately.

On Figure 3.28 the graph showing the function $\Delta \overline{m}_{x}(\overline{\omega}_{x})$ is plotted for the entire range of angular rolling velocities. For type E rolling maneuvers we can define five basic regions of values

Fig. 3.26

Fig. 3.27

Type E Rolling Maneuver
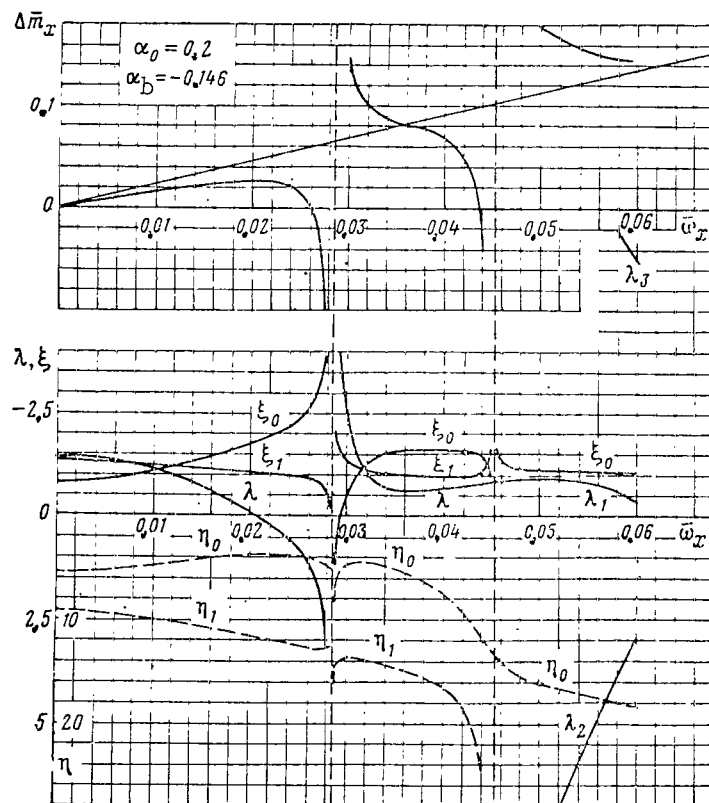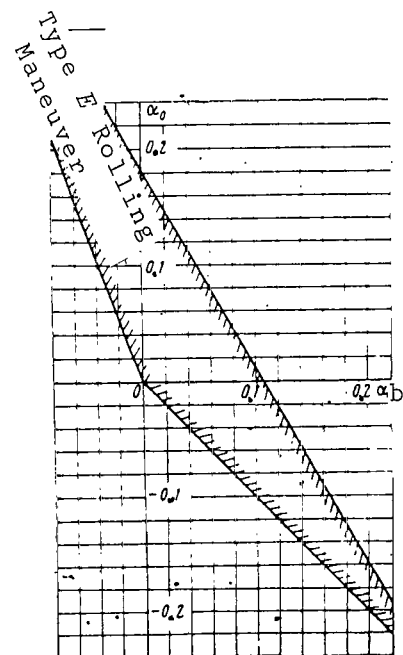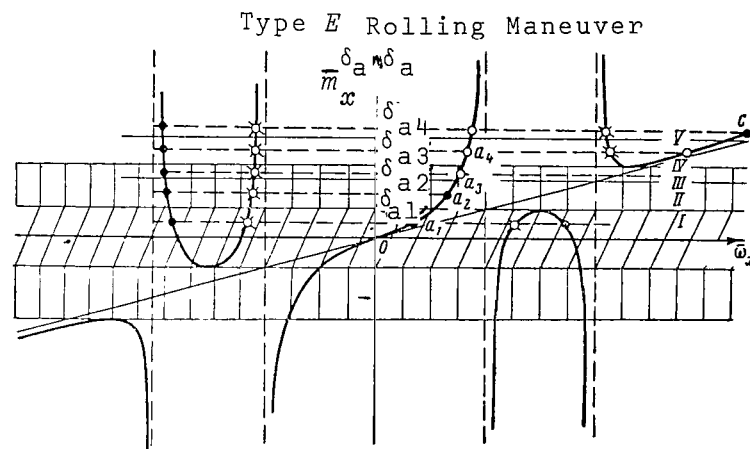
Fig. 3.28

of the quantity $\Delta\overline{m}_x(\overline{\omega}_x)$ which differ in number and type of singular points (see Fig. 3.28 and 3.29).

First region

$$|\Delta\overline{m}_x| \leqslant |\Delta\overline{m}_{\lambda 1}|. \tag{3.69}$$

In the first region there are five singular points, three singular points of stable spatial-focus type and two singular points of spatial-saddle-type which correspond to the unstable motion and separate the regions of stable motion. The point of "attraction", i.e., the singular point to which the parameter of the motion of an aircraft tends, depends on the initial conditions of motion. for maneuvers which begin from small angular velocities $\overline{\omega}_x(0)$, $\overline{\omega}_y(0)$, $\overline{\omega}_z(0)$ such a point of attraction is the singular point with the least value of the angular rolling velocity $\overline{\omega}_x$. The separatrix
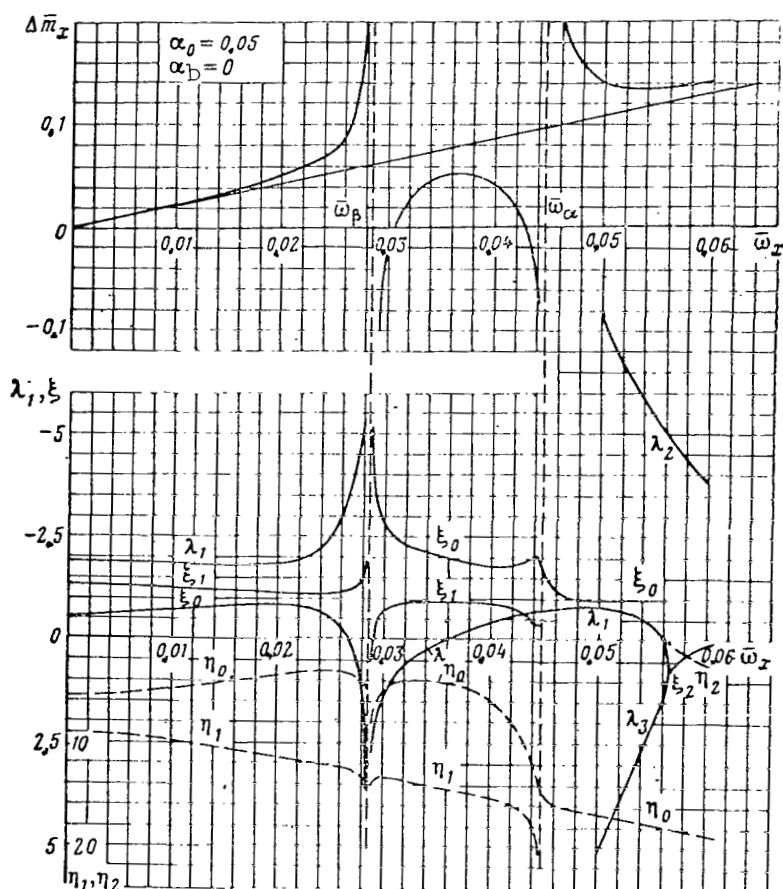
Fig. 3.29

surfaces which leave from the singular points of "saddle" type separate the regions of initial conditions into three, each of which has its own singular point that appears as the point of attraction. An example of the transient condition with gradual deflection of the ailerons when motion is realized in the vicinity of the singular point of a type (see Fig. 3.28) is shown on Figure 3.30. Such controlled motion of an aircraft is one of the most typical for rolling maneuvers and will be studied in greater detail in Chapter IV.
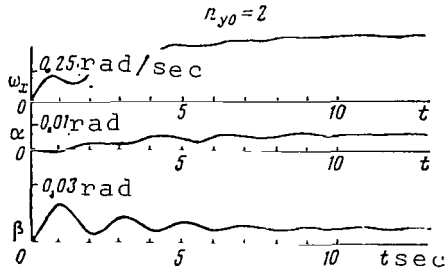
Fig. 3.30

Second region

$$|\Delta \overline{m}_{x1}| < |\Delta \overline{m}_x| \leqslant |\Delta \overline{m}_{x2}|. \tag{3.70}$$

In the second region there are three singular points: two singular points of stable focus type and one of saddle type. The motion during rolling maneuvers, accomplished from undisturbed flight, is realized in the vicinity of the singular point a_2 (see Fig. 3.28), and the transient condition is of the same type as the process shown on Figure 3.30.
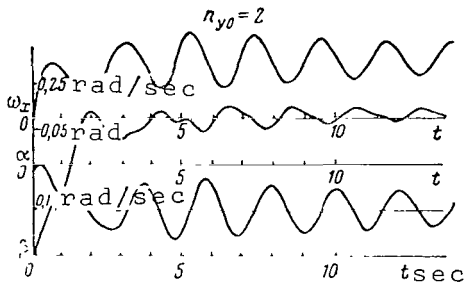
Third region

$$|\Delta \overline{m}_{x2}| \leqslant |\Delta \overline{m}_x| \leqslant |\Delta \overline{m}_{x3}|. \tag{3.71}$$

Fig. 3.31

The number of singular points in the third region is the same as in the second; however, one of the singular points of stable focus type becomes an unstable focus type. With deflections of the ailerons that satisfy conditions (3.71), nonlinear undamped oscillations relative to the unstable focus a_3 are established (see Fig. 3.28). An example of the transient condition corresponding to motion in the vicinity of the singular point of a_3 type is shown on Figure 3.31.

Fourth region

$$|\Delta \overline{m}_{x3}| \leqslant |\Delta \overline{m}_x| \leqslant |\Delta \overline{m}_{x4}|. \tag{3.72}$$

In the fourth region there are two singular points of unstable focus type, one of stable focus type and two saddle-type singular points. The deflection of the ailerons with zero initial conditions leads to the onset of nonlinear undamped oscillations in the vicinity of the singular point a_4 (see Fig. 3.28). The change in the

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parameters of motion of the aircraft in time is analogous to that shown in Figure 3.31.

Fifth region

$$|\Delta \overline{m}_{x4}| \ll |\Delta \overline{m}_x|.$$

(3.73)<u>/121</u>

Motion in the fifth region is analogous to that in the fourth region. The distinction consists in the fact that on the branch of the static curve c (see Fig. 3.28) the singular points of unstable focus type convert to singular points of stable focus type. In certain instances, with gradual deflections of the ailerons, motion relative to the singular point c may be realized. Usually this is associated with the small value of the lateral stability m_x^β, when the aircraft is able to accelerate below large rolling velocities prior to which there arises a rather large retarding torque relative to the axis OX_1. Motion in the vicinity of the singular point a_5 is analogous to that shown on Figure 3.31.

16. Motion of an Aircraft at Large Angular Velocities of Rotation About the Longitudinal Axis

As was shown above we can discern three basic regions of values of the angular rolling velocity $\overline{\omega}_x$ - small, intermediate and large, in each of which the motion of the aircraft has its own characteristics. At small angular rolling velocities, in the equations of motion we can drop the inertial terms and look at them as linear equations taking into account if necessary only the nonlinearity of the aerodynamic coefficients. The most complex rules involve controlled motion of an aircraft accompanied by angular rolling velocities for which the inertial and aerodynamic moments have an identical order of magnitude; this is the region of angular rolling velocities which is studied mainly in the present book. And finally, there is a region of very large angular rolling velocities which is characterized by the fact that the inertial moments of the gyroscopic stability of an aircraft with motion having such angular velocities exceed the moments of the aerodynamic stability. Such a motion is in a certain sense the limiting motion of the aircraft with a rolling maneuver and the results obtained in its investigation assist in the analysis of certain types of motion of an aircraft at large angular rolling velocities.

Large values of the angular rolling velocity $\overline{\omega}_x$ may be assumed to be those for which the following inequality is satisfied

$$\overline{\omega}_x \gg \max(\overline{\omega}_\alpha, \ \overline{\omega}_\beta). \tag{3.74}$$

If the expression for the larger critical angular rolling velocity in explicit form is substituted into inequality (3.74) it is easy to see that this inequality will permit estimating the

relationship between the values of the aerodynamic and inertial moments. In fact, let the larger critical rolling velocity be the value $\overline{\omega}^\alpha$, then inequality (3.74) can be rewritten in dimensional /122 form in the following manner:

$$(J_y - J_x)\,\omega_x^2 \gg -\, m_z^z q S b_A.\tag{3.75}$$

The quantity in the left-hand side of the inequality characterizes the inertial moment, and that in the right-hand side the aerodynamic moment, which act on the aircraft. Inequality (3.75) means that the basic influence on the motion of the aircraft will be that of the inertial moments in comparison with which the effect of the aerodynamic moments of stability is small and can be ignored. Taking into account the smallness of the aerodynamic moments of stability in comparison with the inertial moments, using the formulas from Table 2, it is easy to show that the static solutions tend toward the following limiting values when $\overline{\omega}_x \to \infty$:

$$\left. \begin{aligned} & \lim_{\overline{\omega}_x \to \infty} (\alpha_0 + \alpha_{s})_s \to 0; \\ & \lim_{\overline{\omega}_x \to \infty} \beta_{cT} \to 0; \quad \lim_{\overline{\omega}_x \to \infty} \overline{\omega}_{y\,s\,s} = \lim_{\overline{\omega}_x \to \infty} \overline{\omega}_{z\,s\,s} \to 0. \end{aligned} \right\} \tag{3.76}$$

In this case, relationship (3.76) is satisfied for any deflections of the elevator (δ_e) and the rudder (δ_r), which is explained by the large gyroscopic stability of motion of the aircraft.

Substituting Expression (3.76) into the equations of motion of the aircraft (1.33) as parameters of the undisturbed motion, let us linearize them and find the equations in variations relative to the motion including the rapid rotation of the aircraft relative to the longitudinal axis at an angular rolling velocity $\overline{\omega}_x = \Omega$:

$$\left. \begin{aligned} & \overline{\omega}_z' + \mu A \Omega \overline{\omega}_y = \overline{m}_z^{\overline{\omega}_z}\, \overline{\omega}_z; \\ & \overline{\omega}_y' - \mu B \Omega \overline{\omega}_z = \overline{m}_y^{\overline{\omega}_y}\, \overline{\omega}_y; \end{aligned} \right\} \tag{3.77}$$

$$\left. \begin{aligned} & \alpha' + \mu \Omega \beta + \frac{c_y^\alpha}{2}\, \alpha = \mu \overline{\omega}_z; \\ & \beta' - \mu \Omega \alpha - \frac{c_z^\beta}{2}\, \beta = \mu \overline{\omega}_y; \end{aligned} \right\} \tag{3.78}$$

$$\overline{\omega}_x' - \overline{m}_x^{\overline{\omega}_x}\, \overline{\omega}_x = \overline{m}_x^\beta \cdot \beta.\tag{3.79}$$

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In Equations (3.77)-(3.79) all the variables are variations relative to their own nominal values

$$\bar{\omega}_x = \Omega; \quad \bar{\omega}_{y\,ss} = \bar{\omega}_{z\,ss} = \alpha_0 + \alpha_{ss} = \beta_{ss} = 0. \tag{3.80}$$

Equations (3.77)-(3.79) may be integrated in explicit form. It is easy to see that the equations in practice are divided into two pairs of linear equations of second order and one equation of first order, which may be solved sequentially. Solution to Equations (3.77) does not depend on the remaining equations and may be written in the following form:

$$\left. \begin{aligned} \bar{\omega}_y &= A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t}; \\ \bar{\omega}_z &= B_1 e^{\lambda_1 t} + B_2 e^{\lambda_2 t} \cdot \end{aligned} \right\} \tag{3.81}$$

The complex-conjugate roots λ_1, λ_2 can be conveniently determined by using the approximate formula based on the assumption of smallness of the coefficients of damping in comparison with the imaginary part of the root

$$\lambda_{1,2} \simeq \frac{1}{2}\left(\overline{m}_z^{\omega_z}\text{b} + \overline{m}_y^{\omega_y}\right) \pm i\mu\Omega \sqrt{AB}. \tag{3.82}$$

The remaining roots of the system of Equations (3.77)-(3.79) with analogous assumptions may be approximately written in the form

$$\lambda_{3,4} \simeq \frac{1}{2}\left(-\frac{c_y^\alpha}{2} + \frac{c_z^\beta}{2}\right) \pm i\mu\Omega; \tag{3.83}$$

$$\lambda_5 = \overline{m}_x^{\omega_x}. \tag{3.84}$$

Equations (3.78) and (3.79) are nonuniform linear equations relative to the variables α, β and $\bar{\omega}_x$, since we may assume that the solutions for $\bar{\omega}_y$ and $\bar{\omega}_z$ are known. Solutions for α and β depend on the change in the variation of the angular velocities $\bar{\omega}_y$ and $\bar{\omega}_z$ in the process of motion, thanks to which the solution for the variations of the angular rolling velocity $\bar{\omega}_x$ also depends on the variation of $\bar{\omega}_y$ and $\bar{\omega}_z$. Thus the solution for $\bar{\omega}_y$ and $\bar{\omega}_z$ depends on two roots of the characteristic equation and two constants which are determined by the initial conditions; the solution for α and β in the general case of arbitrary initial conditions depends on four roots and four constants and the solution for $\bar{\omega}_x$ already depends on all five roots and five constants. This partial separation of motions in particular comprises the difference between the

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limiting case of rotation of an aircraft at a high angular rolling velocity and the general case of motion when the solution to the linearized equations for all parameters of motion depend on the five roots and the five constants. Further simplification of analysis of the disturbed motion of an aircraft may be obtained if we analyze the case when $\overline{m}_x^\beta = 0$. In satisfying this condition, the solutions to the equations for variations in the angular velocities $\overline{\omega}_x$, $\overline{\omega}_y$, and $\overline{\omega}_z$ do not depend on the angles of attack and sideslip.

Let us analyze this case in greater detail by looking at motion in phase space of angular velocities. In Equation (3.79) we find the general solution for $\overline{\omega}_x$:

$$\overline{\omega}_x = \overline{\omega}_{x0} e^{\lambda_s \tau}. \tag{3.85}$$

Substituting into solution (3.81) expressions for the roots let us transform the relationship for $\overline{\omega}_y$ and $\overline{\omega}_z$ to the form

$$\left.\begin{aligned}
\overline{\omega}_y e^{-\frac{1}{2}\left(\overline{m}_z^{\overline{\omega}_z} + \overline{m}_y^{\overline{\omega}_y}\right)\tau} &= c_{11}\sin\left(\mu\Omega\sqrt{AB}\right)\tau; \\
\overline{\omega}_z e^{-\frac{1}{2}\left(\overline{m}_z^{\overline{\omega}_z} + \overline{m}_y^{\overline{\omega}_y}\right)\tau} &= c_{21}\sin\left(\mu\Omega\sqrt{AB}\right)\tau + c_{22}\cos\left(\mu\Omega\sqrt{AB}\right)\tau.
\end{aligned}\right\} \tag{3.86}$$

From Expression (3.86), by carrying out simple computations, we find

$$\left.\begin{aligned}
\frac{\overline{\omega}_y e^{-\frac{1}{2}\left(\overline{m}_z^{\overline{\omega}_z} + \overline{m}_y^{\overline{\omega}_y}\right)\tau}}{c_{11}} &= \sin\left(\mu\Omega\sqrt{AB}\right)\tau; \\
\frac{\left(\overline{\omega}_z - \frac{c_{21}}{c_{11}}\overline{\omega}_y\right) e^{-\frac{1}{2}\left(\overline{m}_z^{\overline{\omega}_z} + \overline{m}_y^{\overline{\omega}_y}\right)}}{c_{22}} &= \cos\left(\mu\Omega\sqrt{AB}\right)\tau.
\end{aligned}\right\} \tag{3.87}$$

Let us introduce new variables which relate the linear functions and the angular velocities $\overline{\omega}_y$ and $\overline{\omega}_z$:

$$\left.\begin{aligned}
\frac{\overline{\omega}_y}{c_{11}} &= \overline{\omega}_{y1}; \\
\left(\overline{\omega}_z - \frac{c_{21}}{c_{11}}\overline{\omega}_y\right)\frac{1}{c_{22}} &= \overline{\omega}_{z1}.
\end{aligned}\right\} \tag{3.88}$$

The latter transformation is feasible to carry out since the motion

in the phase space of the new variables $\bar{\omega}_{y1}$, $\bar{\omega}_{z1}$, $\bar{\omega}_x$ is described by a relationship which is simpler than in the original phase space. Let us square each of the relationships (3.87) and add them. Carrying out the substitution of variables in Equations (3.88) we find the equations of the integral curves in phase space for the variables $\bar{\omega}_{y1}$ and $\bar{\omega}_{z1}$:

$$\bar{\omega}_{y1}^2 + \bar{\omega}_{z1}^2 = e^{\left(\bar{m}_z^{\bar{\omega}_z} + \bar{m}_y^{\bar{\omega}_y}\right)\tau}. \tag{3.89}$$

For each value of the moment of time τ (3.89) is the equation of /125 the circumference. If we exclude the time from Equation (3.89) and use the solution for $\bar{\omega}_x$ (3.85), we find the equation for the family of surfaces in phase space on which the following integral curves lie.

$$\bar{\omega}_{y1}^2 + \bar{\omega}_{z1}^2 = c_0 \left(\bar{\omega}_x\right)^{\frac{\bar{m}_x^{\bar{\omega}_x}}{\bar{m}_z^{\bar{\omega}_z} + \bar{m}_y^{\bar{\omega}_y}}}; \tag{3.90}$$

$$c_0 = (\omega_{x0})^{-\frac{\bar{m}_x^{\bar{\omega}_x}}{\bar{m}_z^{\bar{\omega}_z} + \bar{m}_y^{\bar{\omega}_y}}}. \tag{3.91}$$

For each value of the coefficient c_0 the surface which is described by Equation (3.90) is a rotational body relative to the axis $O\bar{\omega}_x$. If the solutions for $\bar{\omega}_{y1}$, $\bar{\omega}_{z1}$, and ω_x are stable, then the motion in phase space occurs over the surface of the parabaloid which is tangent to the origin.
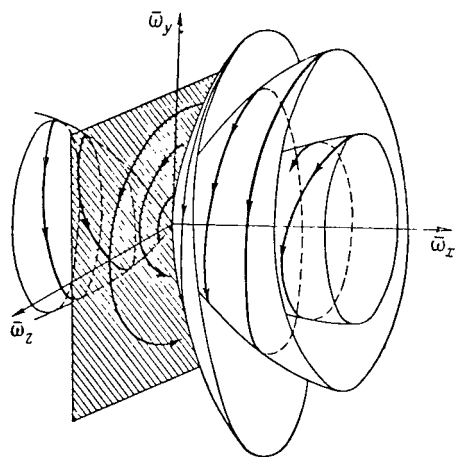
$$\bar{\omega}_x = \bar{\omega}_{y1} = \bar{\omega}_{z1} = 0. \tag{3.92}$$

The diagram of the motion in phase space is shown in Figure 3.32.

The system of equations of motion (3.77)-(3.79) has one real root of Equation (3.84) which determines the separatrix surface, the equation of which immediately follows from the solution for $\bar{\omega}_x$ in Expression (3.85).

Fig. 3.32

$$\overline{\omega}_x = 0. \tag{3.93}$$

From Equation (3.93) it follows that the separatrix surface is
a plane of the orthogonal axis $O\overline{\omega}_x$. The surfaces which are des-
cribed by Expression (3.90), on which the integral curves are lo-
cated, with an increase in the value of c_0 tend toward the plane
with Equation (3.92) but never reach it. When one of the roots
$\lambda_{3,4}$ or λ_5 has a positive real part, the motion is unstable and the
generatrix of the surface which is described by Equation (3.90) has
singularity in the vicinity of the singular point, agreeing with
the origin (Fig. 3.33).
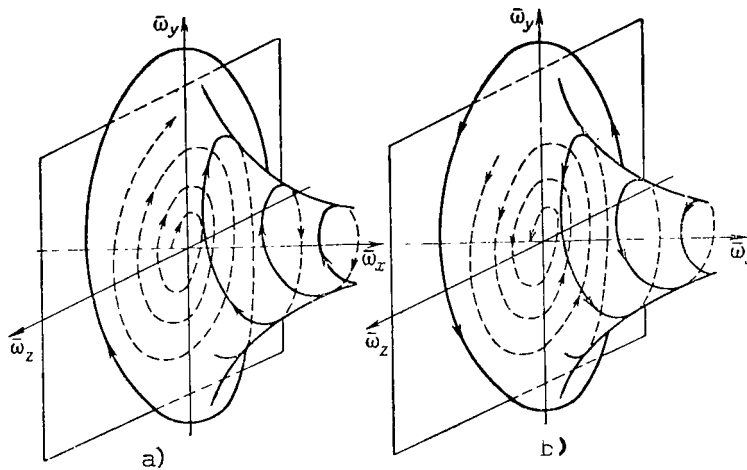
Fig. 3.33

In this case the motion in the separatrix surface will be stable if
$\overline{m_x^{\overline{\omega}}}x < 0$ (see Fig. 3.33,a) and unstable if $\overline{m_x^{\overline{\omega}}}x > 0$ (see Fig. 3.33,b).
Motion in the phase space of the real angular velocities ω_y, ω_z,
and ω_x has the same characteristics as the motion in space of the
variables ω_{y1}, ω_{z1}, $\overline{\omega}_x$. The difference consists only in the fact
that, if in the space ω_{y1}, ω_{z1}, ω_x, the cross section of the surface
along which the figurative point with the plane of the orthogonal
axis $O\overline{\omega}_x$ moves, is a circle then in the space ω_y, ω_z, ω_x, such
cross section is an ellipse. If the aircraft in its inertial and
aerodynamic characteristics is symmetric relative to the axes OY
and OZ, then it is easy to show that the coefficient c_{21} in rela-
tionship (3.86) will be equal to zero and the ellipse is degenerated
into a circle.

Now let us proceed to an analysis of the changes in the varia-
tions of the angles of attack and side slip of the aircraft with
rapid rolling rotation. Change in the angles of attack and side
slip of an aircraft in disturbed motion is described by Equation
(3.78). Let us look in first order at the motion of an aircraft
in the presence of initial disturbances only in the angles of attack

and side slip, i.e., when $\overline{\omega}_y(0) = \overline{\omega}_z(0) = 0$. Such a disturbance can be represented as the effect on the aircraft of a gradual wind blast of large extent. From expressions (3.77) it follows that the angular velocities $\overline{\omega}_y$ and $\overline{\omega}_z$ remain identically equal to 0, i.e., the longitudinal axis of the aircraft in disturbed motion also retains a constant orientation relative to inertial space and may be shifted only by plane-parallel motion. Retention of the invariable orientation of the longitudinal axis of the aircraft with respect to inertial space is explained by the large degree of gyroscopic stability.

Since $\overline{\omega}_y = \overline{\omega}_z = 0$, Equations (3.78) may be rewritten in a more simple form.

$$\left.\begin{array}{l} \alpha' + \mu\Omega\beta + \dfrac{c_y^\alpha}{2}\,\alpha = 0; \\[3mm] \beta' - \mu\Omega\alpha - \dfrac{c_z^\beta}{2}\beta = 0.. \end{array}\right\} \tag{3.94}$$

For further analysis we have the equation for the angle of bank of an aircraft

$$\gamma' = \mu\Omega. \tag{3.95}$$

In the specific case when

$$c_y^\alpha = -c_z^\beta = 0, \tag{3.96}$$

the solution to Equation (3.94) has an especially simple form

$$\left.\begin{array}{l} \alpha = \varphi_0\cos\gamma; \\ \beta = \varphi_0\sin\gamma. \end{array}\right\} \tag{3.97}$$

Let us note that the direct proof is easy to find that the solution to (3.97) satisfies the system of Equations (3.94) with the arbitrary function $\Omega(t)$.

From solution (3.97) it is clear that changes in the angle of attack and side slip of an aircraft are caused by kinematic relationships (Fig. 3.34), but the angle between the longitudinal axis of the aircraft and the velocity vector ϕ^* remains constant.

$$\varphi^* = \sqrt{\alpha^2 + \beta^2} = \varphi_0. \tag{3.98}$$

In the general case when Equation (3.96) is not satisfied the solution to Equation (3.94), with the proper choice of the beginning of the time readings, can be written in the following form:

$$\alpha = \varphi_0 e^{-\frac{1}{2}\left(\frac{c_y^\alpha}{2} - \frac{c_z^\beta}{2}\right)\tau} \cdot \cos \gamma;$$

$$\beta = \varphi_0 e^{-\frac{1}{2}\left(\frac{c_y^\alpha}{2} - \frac{c_z^\beta}{2}\right)\tau} \cdot \sin(\gamma + \varphi_1) \tag{3.99}$$
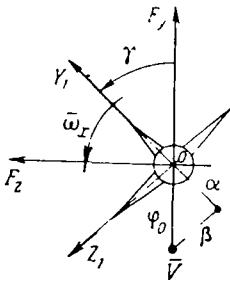
Fig. 3.34

From expressions (3.99) it is clear that the solution for α and β is an oscillational damped process. The oscillational character in the change of the angles α and β is due to their kinematic relationship with ϕ^*, i.e., the angle between the longitudinal axis of the aircraft and the velocity vector. Furthermore due to the effect of aerodynamic lift the aircraft begins to "drift" along the current and the angle ϕ^* between the fixed axis of rotation of the aircraft and the velocity vector in inertial space is decreased.

Let us look at the motion of an aircraft in more detail. We determine the projections of the forces which act on the aircraft for the axes OF_y and OF_z that are associated with the center of mass and are shifted in a plane-parallel direction along with the aircraft (see Fig. 3.34):

$$\left. \begin{array}{l} F_y = Y_1 \cos \gamma - Z_1 \sin \gamma; \\ F_z = Y_1 \sin \gamma + Z_1 \cos \gamma, \end{array} \right\} \tag{3.100}$$

where Y_1 and Z_1 are the projections of the aerodynamic forces on the body axes of the aircraft.

With an accuracy up to a constant factor, from Equations (3.99) we find expressions for the projection of forces on the body axes of the aircraft OY_1 and OZ_1 in the following form:

$$\left. \begin{array}{l} Y_1 = c_y^\alpha \alpha = A_0^* \cos \gamma \cdot e^{-\frac{1}{2}\left(\frac{c_y^\alpha}{2} - \frac{c_z^\beta}{2}\right)\tau}; \\ \\ Z_1 = c_z^\beta \beta = [A_1^* \sin \gamma + A_2^* \cos \gamma] e^{-\frac{1}{2}\left(\frac{c_y^\alpha}{2} - \frac{c_z^\beta}{2}\right)\tau} \end{array} \right\} \tag{3.101}$$

If we substitute expressions (3.101) into Equations (3.100), we find:

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$$F_y = [A_0^* \cos^2 \gamma - A_1^* \sin^2 \gamma - A_2^* \sin \gamma \cos \gamma] e^{-\frac{1}{2}\left(\frac{c_y^\alpha}{2} - \frac{c_z^\beta}{2}\right)\tau} ;$$

$$F_z = [A_0^* \cos \gamma \sin \gamma + A_1^* \cos \gamma \sin \gamma + A_2^* \cos^2 \gamma] e^{-\frac{1}{2}\left(\frac{c_y^\alpha}{2} - \frac{c_z^\beta}{2}\right)\tau} \qquad (3.102)$$

From Expressions (3.102) it is clear that the terms, which are <u>/129</u> in the bracket in the expressions for F_y and F_z, are periodic functions of the angle of bank γ with the period π. The most simple form of Formula (3.102) is obtained in that case when the carrier properties of the surfaces of an aircraft are identical in the planes $X_1 O Y_1$ and $X_1 O Z_1$ (see Fig. 1.2), i.e., the following equations is satisfied.

$$c_y^\alpha = - c_z^\beta. \qquad (3.103)$$

We can show that under the condition of (3.103) the relationships

$$A_0^* = - A_1^*;$$
$$A_2^* = 0. \qquad (3.104)$$

are valid, whence we can obtain directly

$$F_y = A_0^* e^{-c_y^\alpha \tau}; \quad F_z = 0. \qquad (3.105)$$

From relationships (3.105) it follows that in this case the motion of an aircraft occurs in such a way that the increase in the angle of attack which arises after the effect of the disturbance decreases exponentially with time. Physically this can be explained by the "drift" of the rotating aircraft along the direction of the influence of the wind disturbance with the retention of an invariable orientation of its axis of rotation relative to inertial space.
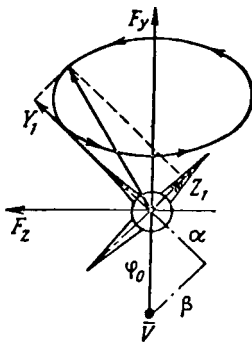
Fig. 3.35

For an aircraft with an arbitrary relationship between the derivative c_y^α and c_z^β, changes in the functions in the brackets of Formulas (3.102) are somewhat more complex and may be illustrated on Figure 3.35. The aerodynamic force which acts on the aircraft in this case, changes in value and direction. With respect to such a variable character of the effect of the aerodynamic force, the motion of a rapidly

rotating aircraft in the general case acted on by a constant wind, directed orthogonal to the axis of rotation of the aircraft will be curvilinear, unlike the rectilinear "drift" along the current of a symmetric missile.

Specific cases of motion of a rapidly rotating aircraft show that two types of motion are observed, the occurrence of which depends on the effective disturbances. If the disturbances act on a rapidly rotating aircraft based on the angles of attack and side slip, then its disturbing motion is reduced to such a change for which the longitudinal axis of the aircraft moves in a plane-parallel direction and does not change its angle of orientation. $\underline{/130}$ If the disturbances lead to a change in the angular velocity then the axis OX_1 of the aircraft begins to precess relative to the vector of the flying speed. Such cases are analyzed in greater detail in Chapter VII relative to the motion of a symetric rotating missile. It should be noted that the problem on the motion on a rapidly rotating aircraft relative to the longitudinal axis is interesting not only as an extreme case of the maneuver of an aircraft with aileron control but also has direct application to the analysis of the dynamics of a rapidly rotating missile with aerodynamic and inertial nonsymmetry.

CHAPTER IV

DYNAMIC CHARACTERISTICS OF AN AIRCRAFT WITH SIMULTANEOUS CONTROL BY AILERONS AND ELEVATOR

Above we obtained the basic relationships which permit finding the values of the parameters of motion of an aircraft representing steady motions. These relationships permit analyzing and obtaining basic qualitative concepts as to the motion of an aircraft in carrying out specific maneuvers. In the present chapter we study the properties and the characteristics of an aircraft with simultaneous control relative to the lateral and longitudinal axes (δ_a, δ_e). Chapter V is devoted to analyzing the motion of an aircraft with simultaneous control relative to the latitudinal and vertical axes (δ_a, δ_r). In each of the chapters we analyze the following basic questions:

1. The physical picture of the motion of an aircraft is analyzed.

2. The characteristics and differences in spatial motion of an aircraft are determined in comparison with isolated motions, and their causes are analyzed.

3. The stability of motion of an aircraft is analyzed both in the process of carrying out the maneuver and with the placement of the controls into a neutral position (with cessation of the maneuver).

4. The dynamics of an aircraft are studied in carrying out a maneuver, and the G-forces which act on the aircraft are evaluated.

From this last problem the basic result can be obtained by numerical calculation on simulators or on digital computers.

The dynamics of an aircraft during rolling maneuvers depends substantially on the relationship between the critical rolling velocities corresponding to a yawing motion ($\bar{\omega}_\beta$) and a pitching motion ($\bar{\omega}_\alpha$). For the majority of modern aircraft it is characteristic that at subsonic flying speeds $\bar{\omega}_\beta \gg \bar{\omega}_\alpha$, and at supersonic speeds, the sign of the equation changes to the opposite ($\bar{\omega}_\beta \ll \bar{\omega}_\alpha$). Below we look at both of these cases.

Even from the results obtained in Chapter III it is obvious what a substantial influence is exerted on the rolling maneuvers of an aircraft by the excess lateral stability. In analyzing specific rolling maneuvers, especially at subsonic flying speeds, in certain instances the dependence of the lateral stability on the angle of attack $m_x^\beta(\alpha)$ exerts a qualitative influence on the dynamic characteristics of the aircraft. In this respect in each of the chapters we carry out an analysis of the influence of this function.

17. Characteristics of Rolling Motion of an Aircraft with Simultaneous Control by Ailerons and Elevator

In studying the type of singular points, we found that as a function of the parameters characterizing the control of the aircraft in the longitudinal plane [of its angle of attack at the beginning of a maneuver (α_0) and of the increase in the angle of attack during pitching ($\Delta\alpha_b$)] the singular point to which the parameters of motion of the aircraft tend, in proportion to damping of the transient condition, will lie in the phase space either nearer to the axis $\bar{\omega}_x = 0$ than does the first critical angular rolling velocity or beyond the first critical velocity. From all of the numerous types of rolling maneuvers $A - F$ (see Sections 13 and 15), we can distinguish two basic groups which differ in the character of change of the static curve $\Delta m_x = f(\bar{\omega}_x)$ in the range of angular velocities less than the first critical.

$$0 \leqslant |\bar{\omega}_x| \leqslant \min\left(\dot{\bar{\omega}}_a,\ \bar{\omega}_\beta\right). \tag{4.1}$$

To the first group belong those types of maneuvers for which all the singular points in the region of angular velocities satisfying inequality (4.1) are represented by a periodically stable solution. These are maneuvers of type C, E, and F (see Section 13 and Fig. 3.12). To the second group belong those maneuvers for which in the range of angular rolling velocities (4.1) there are eight periodic unstable singular points [these are maneuvers of A, B and D types (see Fig. 3.12)].

For maneuvers of the first group the behavior of the aircraft in controlled flight is characterized in first order by the seeming effectiveness of the ailerons, in proportion to the approach of the angular rolling velocity of the aircraft to a critical value, beginning to drop, and significant increases in the aileron deflections lead only to a small increase in the value of the angular rolling velocity ω_x (Fig. 4.1). The type of transient conditions for the basic parameters of motion of an aircraft with a gradual deflection of the ailerons carried out from the conditions of flight with the G-force $n_y = 2$ is shown on Figure 4.2. The parameters of motion of an aircraft tend, in proportion to damping of the oscillations, toward a certain steady value (ω_{xss}, α_{ss}, β_{ss}, etc), which are determined by the static solutions, and the transient conditions of the basic variables have a significant overshoot relative to the

steady values. In carrying out maneuvers of the first group the
rolling motion of an aircraft is accompanied by large changes in
the angles of side slip and attack, and significant G-forces may
affect the pilot and the design of the aircraft.

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The basic property of maneuvers of the first group, the seeming
decrease in effectiveness of the ailerons with increase in angular
rolling velocity $\bar{\omega}_x$, is explained by the simultaneous effect on the
dynamics of the aircraft of the lateral stability (m_x^β) and the in-
ertial cross couplings. Such an influence can be explained with
the help of the following simple discussion. Due to the inertial
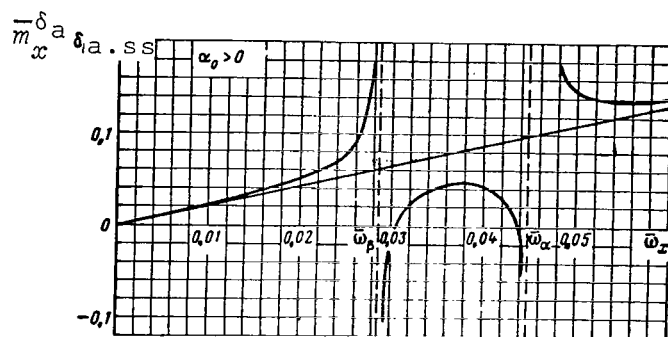
Fig. 4.1.

interaction of the longitudinal and lateral motions the effective
degree of the static stability of an aircraft with motion at an
angular velocity $\bar{\omega}_x$ seems to decrease, as a result of which con-
ditions are created for the development of an angle of side slip.
In turn, the angle of side slip, due to the presence of lateral
stability ($m_x^\beta < 0$) leads to the appearance of a rolling moment which
is built up with the rolling moment from the ailerons and affects
the size of the angular rolling velocity $\bar{\omega}_x$; when the axis of the
aircraft in the beginning of the maneuver is located above the veloc-
ity vector, such an influence appears in the deceleration of rota-
tion of the aircraft. This property of the rolling maneuver is
characteristic both for flight at subsonic speeds when $\omega_\alpha \ll \omega_\beta$, and
for flight at supersonic speeds when $\omega_\alpha \gg \omega_\beta$. Especially strongly
does the seeming limitation of the effectiveness of the ailerons
appear with the relationship of the critical rolling velocities
when $\omega_\alpha \ll \omega_\beta$, i.e., at supersonic speeds. This is due to the fact
that with such a relationship of the critical velocities the air-
craft has a lesser stability in yawing and with a rolling maneuver
can easily escape to large angles of side slip β. Let us look in
somewhat greater detail at the motion of an aircraft with deflection
of ailerons from the conditions of horizontal flight. At the be-
ginning of the transient condition after the ailerons are deflected,
the rotation of the aircraft occurs relative to the major inertial
axis OX_1, as a result of which the aircraft rotates at the angle   /134
of bank γ, its angle of attack α is somewhat decreased and the

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angle of side slip β appears due to the kinematic relationship be-
tween α and β during the roll which may be determined from the ap-
proximate relationship (Fig. 4.3).

Fig. 4.2.

$$\beta \simeq \alpha_0 \gamma. \qquad (4.2)$$

From relationship (4.2) it follows that if the initial value of the angle of attack α_0 is positive, then with rolling the side slip of the aircraft is developed with the same sign as the angular rolling velocity ω_x. With motion of the aircraft with side slip the moment of lateral static stability $\bar{m}_x^\beta$ appears which counteracts its rolling rotation. The process described above can be traced on Figure 4.2, in particular, the decrease in the angle of attack α at the beginning of the transient condition, the development of the angle of side slip, etc. After damping of the transient condition with steady rotation of an aircraft acted on by the deflected ailerons, its axis OX_1 and the vector of total angular velocity retain their mutual position unchanged and move along a cone around the vector of flight velocity (see Fig. 4.3,c), i.e., a regular precession of the aircraft is observed.

For rolling maneuvers of the second group (maneuvers of A, B and C types) we find it to be characteristic that there is a disrup-
tion of the continuous dependence of the angle of rolling velocity
of the aircraft on the angle of aileron deflection with an increase in the aileron deflection (δ_a) greater than a certain value ($\delta_a > \delta_{a1}$; Fig. 4.4). As a result, in the characteristics of the motion of an aircraft during rolling maneuvers we observe substantial changes in the increase of aileron deflection. With small deflec-
tions of the aileron, the aircraft rotates at an angular rolling velocity less than the first critical and retains the usual char-
acteristics of controllability. With relatively large deflections of the ailerons, the angular rolling velocity of the aircraft, due to the effects of side slip at the moment of lateral stability, begins to grow substantially and exceeds the value of the second rolling velocity (Fig. 4.5). In this case, placing the ailerons into a neutral position, or even changing the sign of their deflec-
tion, does not always end the rolling rotation of the aircraft. A practical loss in controllability of the aircraft for the ailerons is observed, the so-called system of "inertial rotation" of the aircraft. The angular rolling velocity, after placing the ailerons

into a neutral position, is retained due to the development of the angle of side slip and the effect on the aircraft of the moment of lateral stability. From the condition of retaining the angular rolling velocity $\omega_x \sim \omega_{x2\ crit}$, it is easy to find the value required for this angle of side slip.

$$\beta \simeq \frac{-\overline{m}_x^{\tilde{\omega}x} \cdot \overline{\omega}_{x2 \cdot} crit}{\overline{m}_x^\beta} .$$
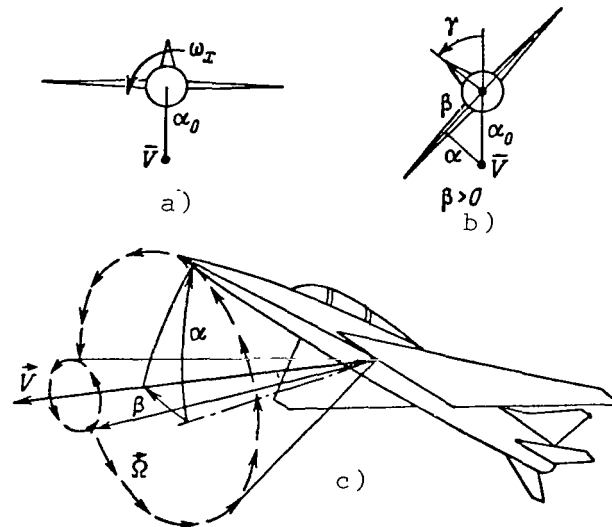
Fig. 4.3.

According to the value of the angle of side slip β, the lateral G-force can be approximately evaluated which affects the aircraft when it enters into the system of inertial rotation.

$$\max |n_z| \simeq \left| k_{n\ z} \cdot \frac{c_z^\beta\ qs}{G} \cdot \frac{\left(-\overline{m}_x^{\tilde{\omega}x}\right) \overline{\omega}_{x\ 2} crit}{\overline{m}_x \beta} \right|$$

In this expression the empirical coefficient k_{IIz} takes into account the dynamics of the transient condition [the overshoot for the angle of side slip ($k_{IIz} \simeq 1.5 - 2$)].

Fig. 4.4

Fig. 4.5

Fig. 4.6

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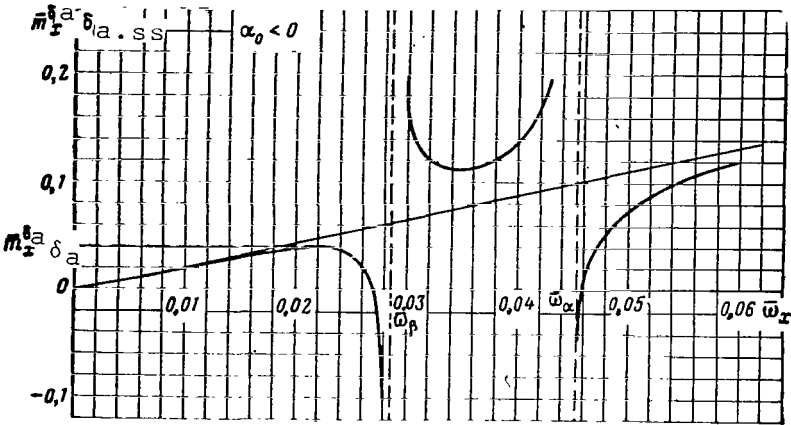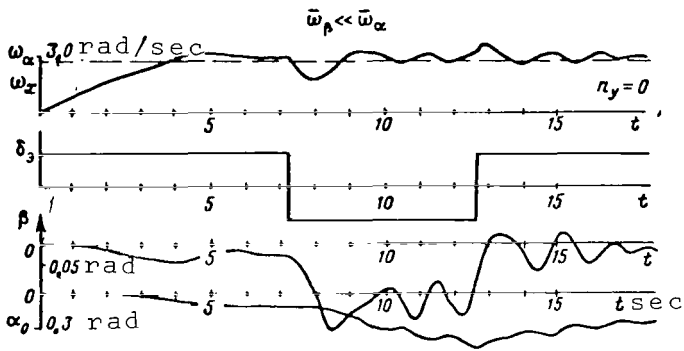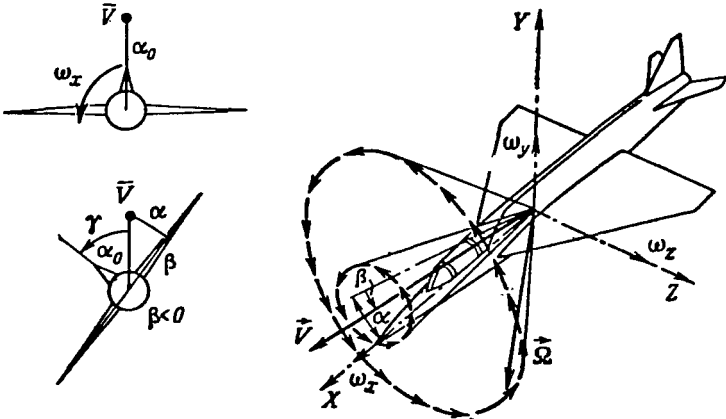
The character of the motion of an aircraft during rolling maneuvers carried out from conditions of flight with a negative G-force ($\alpha_0 < 0$), depends on the simultaneous influence on the simultaneous influence on the dynamics of the aircraft of the inertial cross coupling and the lateral stability m_x^β which in this case is not decelerated, but on the contrary facilitates growth of the angular rolling velocity. The occurrence of the initial side slip, which facilitates growth of the angular rolling velocity, is caused by the kinematic relationship between the angles of attack and side slip during rolling. In fact, as noted above, at the first moment after deflection of the ailerons, rotation of the aircraft occurs relative to the major inertial axis OX_1 and the aircraft rotates at an angle of bank γ, due to which there appears the angle of side slip β, which is determined by approximate relationship (4.2). The angle of side slip β creates a moment relative to the longitudinal axis OX_1:

$$\Delta \overline{M}_x = \overline{M}_x^\beta \cdot \alpha_0 \gamma. \qquad (4.3)$$

If, during the rolling maneuver, the initial angle of attack of the aircraft α_0 is negative the supplementary aerodynamic moment $\Delta \overline{M}_x$ caused by the development of side slip has the same sign as the moment from the aileron. After damping of the transient condition a regular precession of the aircraft is established around the velocity vector $\overline{V}$, in which case the angle between the major inertial axis OX_1 and the vector of total angular velocity of the aircraft remains constant (Fig. 4.6).

The mutual position of the major inertial axis OX_1 of the aircraft and the vector of total angular velocity depends on the size of the angular rolling velocity. If the value of the angular rolling velocity satisfies the inequality $\omega_x < \omega_{1\,crit}$, then the steady value of the angle of attack of the aircraft is negative, if though $\omega_x > \omega_{2\,crit}$, then the angle of attack of the aircraft is positive. In both cases the aircraft, prior to the rolling maneuver, was trimmed at a negative angle of attack. With respect /138 to this latter result (the dependence of the angle of attack on the size of the angular rolling velocity), it is necessary to make a more detailed analysis of the reaction of the aircraft to deflection of the elevator during the rolling maneuvers.

18. Reaction of an Aircraft to Deflection of the Elevator during Rolling Maneuvers

Let us look at the dependence of the reaction of an aircraft, in angle of attack for deflection of the elevator, on the size of the angular rolling velocity during maneuvers. The sought relationship between α and $\Delta \overline{m}_{z\,\delta_e}$ is determined by the static derivative $A_{m\,\delta_e}^\alpha$, the formula for which was given in Table 2. Let us write this relationship and determine its basic properties.

$$A^\alpha_{m\delta_e} = \frac{\mu^2}{A_0}\left(-\overline{m}^\beta_y - \mu\overline{\omega}^2_x B + \frac{c^\beta_z \overline{m}^{\overline{\omega}_y}_y}{2\mu}\right).$$

(4.4)

For simplification, if we ignore the computation of the influence of damping and take into account the expression for the free term of the characteristic equation A_0 (see Table 2) we find

$$A^\alpha_{m\delta_e} = \frac{\alpha_{SS}\,S}{\overline{\Delta m_z}} \simeq \frac{1}{(-\overline{m}^\alpha_{z_b} - A\mu\overline{\omega}^2_x)}.$$

(4.5)

For relationship (4.5) there follows the quite interesting result which includes the fact that the reaction of the aircraft in angle of attack (normal G-force) to deflection of the elevator during the rolling maneuver in first order depends on the relationship between the angular rolling velocity and the critical angle of velocity corresponding to the pitching motion ($\overline{\omega}_\alpha$). In those cases when the value of the angular rolling velocities satisfies the inequality $\overline{\omega}_x < \overline{\omega}_\alpha$, with the rolling maneuvers normal agreement is retained between deflection of the elevator and increase in the angle of attack of the aircraft. In those cases when $\overline{\omega}_x > \overline{\omega}_\alpha$, the sign in the relationship between α_{SS} and $\Delta\overline{m}_z$ is changed to the opposite. The change in the agreement of the signs of α_{SS} and $\Delta\overline{m}_z$ is explained by the fact that the motion of the aircraft which is first trimmed at a positive angle of attack, with an angular rolling velocity exceeding the value of the second critical velocity, occurs with a negative normal G-force and with trim of the aircraft at a negative angle of attack with a positive G-force. Let us look at the reasons for such a change in the relationship between the increase in the angle of attack α_{SS} and $\Delta\overline{m}'_z$. As noted in Section 6, the motion of an aircraft with an angular rolling velocity due to the kinematic interrelationship between α and β in a certain sense is analogous to the motion in the presence of an external periodic effect. When the value of the angular rolling velocity approaches the natural frequency of the longitudinal oscillations of an aircraft, there begins a phenomenon which is similar to resonance. Just as with any resonance the reaction of an object to disturbance at a frequency less than the resonance frequency has a regular sign and at higher frequencies there is a phase lag by 180° and consequently the sign of the reaction changes. An analogous phenomenon is observed also during the rolling maneuver of an aircraft. At this time there is still no angular rolling velocity less than the resonance frequency equal to the critical rolling velocity (ω_α); the reaction of the aircraft, for the angle of attack, to the external moment $\Delta\overline{m}_z$ has a normal sign; with an increase in the angle of velocity ω_x the sign of the increase is changed to the reverse. It is obvious that similar such properties will be possessed by the reaction of the aircraft during the angle of side slip with control by the rudder. (This problem is studied in Chapter V).

In all of the discussions given above the presence in the aircraft of damping has not been taken into account since it somewhat complicates the picture of the motion presented above. In fact, it is easy to prove that due to the presence of damping, in the case when the inequality $\bar\omega_\beta \ll \bar\omega_\alpha$ is satisfied, the free term of the characteristic equation A_0 vanishes at values of the angular rolling velocity $\bar\omega_x$ larger than the numerator of expression (4.4) and when $\bar\omega_\beta \gg \bar\omega_\alpha$ - at smaller values of $\bar\omega_x$. As a result, the form of the function $A_{m\delta_e}^\alpha(\bar\omega_x)$ depends on the relationship between the critical

angular rolling velocities $\bar\omega_\alpha$ and $\bar\omega_\beta$ (Fig. 4.7). On Figure 4.7 the dotted line shows the respective functions for an aircraft not possessing damping. We must additionally note that the reaction of an aircraft for the angular pitching velocity $\bar\omega_z$ to deflection of the elevator retains the regular agreement for all values of the angular rolling velocities. On Figure 4.7 it is obvious that in the case when the inequality $\bar\omega_\alpha \ll \bar\omega_\beta$ is satisfied, the change in the angle of attack of the aircraft during a rolling maneuver grows substantially with an increase in the angular rolling velocity. With such a relationship of the critical velocities the rolling maneuvers of an aircraft are accompanied by large normal G-forces. In the same case when $\bar\omega_\beta \ll \bar\omega_\alpha$, the change in the angle of attack of an aircraft during rolling maneuvers with angular rolling velocities less than the first critical are practically independent of the size of $\bar\omega_x$. In these cases the rolling maneuvers are accompanied by small changes in the normal G-forces.

Thus with the motion of an angular velocity exceeding the second critical which may be valid when the aircraft enters into a system of inertial rotation, to decrease the normal G-forces the pilot must operate the elevator directly opposite to the normal, i.e., with motion involving a positive G-force, to decrease it he must move the control stick toward himself, whereas with a negative G-force he must move it away. It is obvious that such actions are unusual for the pilot and complicate, and sometimes make impossible, piloting the aircraft. An additional complication is the fact that /140 the aircraft under conditions of motion of inertial rotation $(\bar\omega_x > \bar\omega_{2\mathrm{crit}})$ has also a reverse reaction for the angle of sideslip to deflection of the rudder.

Let us look briefly at an evaluation of the angles of side slip which occur during a rolling maneuver with simultaneous deflection of the elevator. The physical reasons for the occurrence of an angle of side slip during a rolling maneuver and deflection of the elevator are due to the gyroscopic properties of a rotating aircraft and are briefly analyzed in Section 6. Analogous to a gyroscope, under the influence of the moment $\Delta\bar{m}_z$ the aircraft precesses in an orthogonal direction. Using Table 2 let us look at the expression for the static derivative $A_{m\delta_e}^\beta$, which relates the

value of the angle of side slip β to the moment of the elevator $\Delta\bar{m}_{z\delta_e}$:

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$$A_{m\delta_e}^{\beta} = \frac{\mu \overline{2\omega}_x}{A_0} \left(B \frac{c_y^\alpha}{2} - \overline{m}_y^{\overline{\omega}_y} \right).$$

(4.6)

Without taking
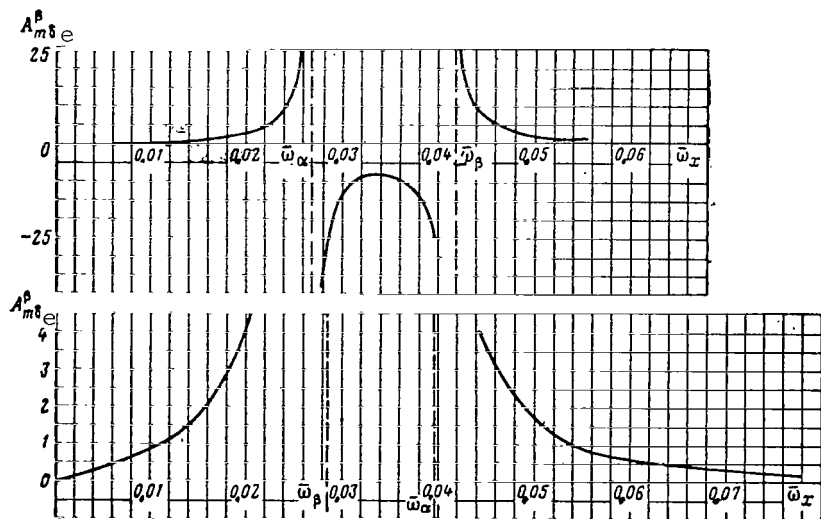damping into
account
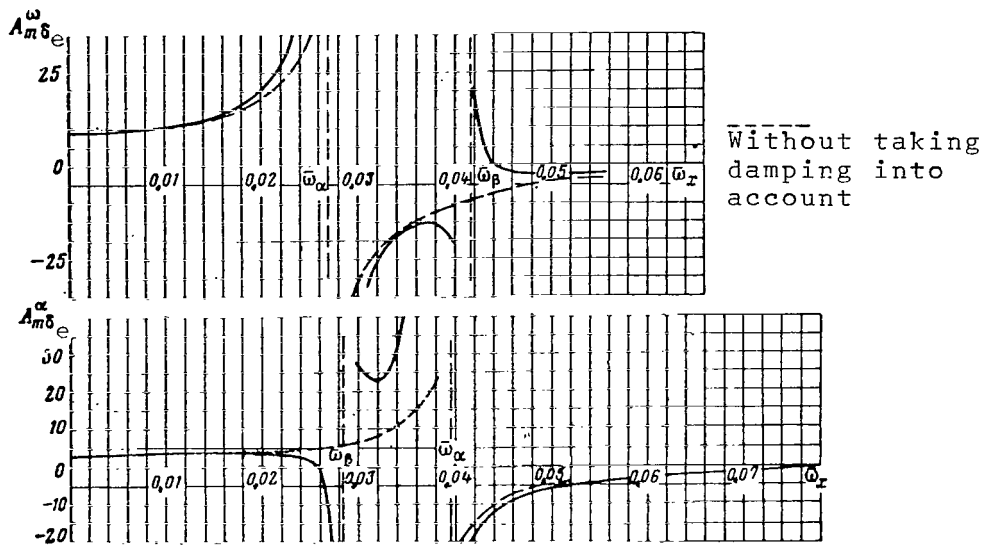
Fig. 4.7.

Fig. 4.8

From relationship (4.6) it follows that between the values $\Delta \overline{m}_z$ and β_{ss} there is a unique relationship that is independent of the relationship between the critical angular rolling velocities of the aircraft (see Fig. 4.8).

An analogous dependence of the increase in the angle of attack α_{ss} and side slip β_{ss} on the value of $\overline{\omega}_x$ is valid if we look at rolling maneuvers carried out from the conditions of horizontal flight with an angle of attack α_0 (see Table 2).

On the basis of the discussion given above we note the following basic rules for the development of angles of attack and side slip of an aircraft during rolling maneuvers. During rolling maneuvers of an aircraft when the inequality $\overline{\omega}_\alpha \gg \overline{\omega}_\beta$ is satisfied (the relationship is characteristic of supersonic flying speeds) and motion occurs at an angular rolling velocity less than the first critical, changes in the normal G-forces (angles of attack α) are small and the basic loads acting on the aircraft are associated with the development of lateral G-forces (angles of side slip β). During rolling maneuvers when $\overline{\omega}_\alpha \ll \overline{\omega}_\beta$ (the relationship is characteristic of subsonic flying speeds), both the angles of attack and side slip may grow substantially, i.e., the normal and lateral G-forces.

It then follows that the reaction of an aircraft for the angle of attack to deflection of the elevator during rolling maneuvers depends only on the size of the deflection of the elevator and the value of the angular rolling velocity, but is independent of the direction of roll (the sign of the angular rolling velocity). This is explained by the fact that the dependence $A^\alpha_{m\delta_e}$ $(A^\alpha_{\alpha 0})$ is an even function of the angular rolling velocity $\overline{\omega}_x$ (see Table 2).

19. Effect of the Dependence of Lateral Stability on the Angle of Attack for Control of an Aircraft by Ailerons and Elevator.

One of the basic assumptions which we find ourselves confront- ing concerning the problems solved above was the assumption as to the linearity of the aerodynamic coefficients of the aircraft. In fact, as we know, an entire series of aerodynamic derivatives of stability of the aircraft depend substantially on the angle of attack. In particular, the derivatives of stability m^α_z, m^β_y, m^β_x, $m^{\delta_e}_x$ may depend substantially on the angle of attack of an aircraft.
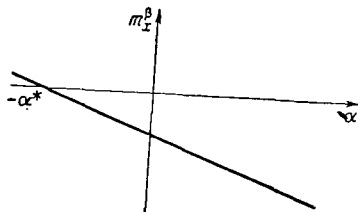
Not all such linear dependences exert a significant influence on the motion of an aircraft during rolling maneuvers, however the effect of several nonlinear dependences of the aerodynamic coefficient may qualitatively change the

Fig. 4.9

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overall picture of its motion and must be taken into account in the
calculation. The most substantial in this respect is taking into
account the nonlinear dependences of the aerodynamic derivatives
which enter into the equation of moments relative to the longitudinal
axis of the aircraft (OX_1). This is caused by the large effect on
the motion of an aircraft of the value of the angle of rolling
velocity. One of the basic aerodynamic characteristics of an air-
craft is the derivative of the moment of the lateral stability for
the angle of side slip m_x^β, which especially at subsonic flying
speeds may substantially depend on the angle of attack of the air-
craft. This dependence in certain instances exerts a qualitative
influence on the characteristic motion of the aircraft during rolling
maneuvers.

Let us look at the dynamic characteristics of an aircraft with
control by the ailerons and elevator in those cases when the deriva-
tive of the moment of the lateral stability m_x^β depends on the angle
of attack. The substantial dependence of the moment of lateral
stability on the angle of attack is especially characteristic for
flight at subsonic speeds; at supersonic speeds it appears consider-
ably more weakly or simply does not occur at all. A typical de-
pendence $m_x^\beta(\alpha)$ for an aircraft with swept wings is shown on Figure
4.9. A characteristic of this dependence is the fact that the
derivative of the lateral stability m_x^β is negative for stable air-
craft; at negative angles of attack the sign changes to the opposite
and becomes positive. Physically this phenomenon is associated
with the characteristics of the manifestation of the forces on a
swept wing and may be explained by the following discussions. Let
us look at a simplified picture of the occurrence of a lateral
moment on an aircraft with a swept wing. With the motion of an
aircraft with side slip (Fig. 4.10), its right and left wings have
different effective sweeps with respect to the current. In this
case the wing on which the aircraft slips has a smaller sweep and
consequently the component normal to the wing of the speed of the
current which creates the basic part of the lift grows. As a result
the size of the lift on the wing with the smaller angle of sweep
with respect to the current grows in comparison to its own mean

Fig. 4.10

value and on the wing with the larger angle of sweep it decreases, thus leading to the appearance of a lateral moment. From the discussions above it follows that the sign of the lateral moment depends on the sign of the mean value of the lift of the wings during side slip. For a positive angle of atttack ($\alpha_0 > 0$) $m_x^\beta < 0$ and for a negative angle of attack ($\alpha_0 < 0$) $m_x^\beta > 0$. The influence of the fuselage, a lateral V-wing and other factors which were not taken into account above somewhat complicate the picture of the manifestation of the lateral moment, however the basic properties of the function $m_x^\beta(\alpha)$ in such case are usually retained.

In the present section, in analyzing the spatial motion of an aircraft the nonlinearity is taken into account only of the aerodynamic moment of the lateral stability $m_x^\beta(\alpha)$, although we know that the other aerodynamic moments and forces in a number of cases depend substantially on the angle of attack. However, these nonlinearities of the coefficients as a rule lead mainly to a quantitative change in the solutions to the equations of motion. In a certain sense the exception consists of aerodynamic moments which enter into the right hand side of the equation for determining the angular rolling velocity. In fact the presence of lateral stability of an aircraft may in some instances lead either to a loss in the stability of motion of an aircraft in rolling, or to the appearance of a stable rotation when the ailerons are placed into neutral position. As was shown above the existence in the aircraft of a /144 moment of lateral stability m_x^β leads to the appearance of substantial differences in the behavior of an aircraft during rolling maneuvers as a function of the conditions of its trim and the longitudinal plane. The dependence of the value m_x^β on the angle of attack α additionally complicates the picture of motion of an aircraft during rolling. With respect to this, in the present section in analyzing the control of an aircraft with ailerons and elevator taking into account the function $m_x^\beta(\alpha)$ the major attention will be paid to investigating the possibilities of loss of controllability of the aircraft involving the ailerons during strong rolling when the angle of rolling velocity exceeds the value of the second critical, i.e, to an analysis of the possibility that systems of inertial rotation will appear. The investigations will be carried out for two cases: the case when the critical angle of rolling velocity corresponding to the pitching motion ($\bar{\omega}_\alpha \ll \bar{\omega}_\beta$) is the smaller, and the case when the smaller critical velocity is determined by the yawing motion $\bar{\omega}_\alpha \gg \bar{\omega}_\beta$). As the results of the investigations carried out below will show, the dynamic characteristics of an aircraft in these two cases are different.

The nonlinear function $m_x(\beta, \alpha)$ will be approximated by the following approximate formula which usually describes the characteristics of the lateral stability of the aircraft quite well.

$$m_x(\beta, \alpha) = (m_{x0}^\beta + m_x^{\alpha\beta}\alpha)\beta.$$

Analysis of the possibility that systems of aircraft motion will develop with losses in controllability by the ailerons (systems of inertial rotation) will be given in two stages. First, let us cite criteria which permitted evaluating the possibilities of the development of such motion in a simplified formulation when it is assumed that m_{x0}^β is small ($m_{x0}^\beta \sim 0$) and $m_x(\alpha, \beta) \simeq m_x^{\alpha\beta} \cdot \alpha \cdot \beta$ and then in the next section we look briefly at the possibilities of refining the criteria obtained in the general case of an arbitrary m_{x0}^β.

Characteristics of Control of an Aircraft with Ailerons when $\omega_\alpha \ll \omega_\beta$

The relationship between the values of the critical rolling velocities when the inequality $\bar\omega_\alpha \ll \bar\omega_\beta$ is satisfied, usually corresponds to flight of an aircraft at subsonic speeds. As was shown above, a quite convenient characteristic which permits finding the basic characteristics of controlled flight of an aircraft during spatial maneuvers is the dependence of the required deflection of the ailerons on the size of the angular rolling velocity. Such a dependence in particular permits investigating as to whether the controllability of the aircraft is maintained when the ailerons are placed into neutral position or whether instances are possible when the angular rolling velocity in such cases is not decreased to a zero value. To find the functions $\Delta\bar m_x(\bar\omega_x)$, we used the formulas for the equilibrium of moments relative to the longitudinal axis.

$$\Delta\bar m_x \simeq -\left[\bar m_x^{\bar\omega_x} \cdot \bar\omega_x + \bar m_x^{\alpha\beta}(\alpha_0 + \alpha_{ss})\beta_{ss} + \bar m_{x0}^\beta \beta_{ss}\right]. \qquad (4.7)$$

The values α_{ss}, β_{ss} may be found using the "static derivatives", the formulas for which are given in Table 2, from the following relationships:

$$\alpha_s = A_{\alpha0}^\alpha \cdot \alpha_0 + A_{m\delta_e}^\alpha \cdot \Delta\bar m_{z\delta_e} + A_{m\delta_r}^\alpha \cdot \Delta\bar m_{y\delta_r} \ldots; \qquad (4.8)$$

$$\beta_s = A_{\alpha0}^\beta \cdot \alpha_0 + A_{m\delta_e}^\beta \cdot \Delta\bar m_{z\delta_e} + A_{m\delta_r}^\beta \cdot \Delta\bar m_{y\delta_r}. \qquad (4.9)$$
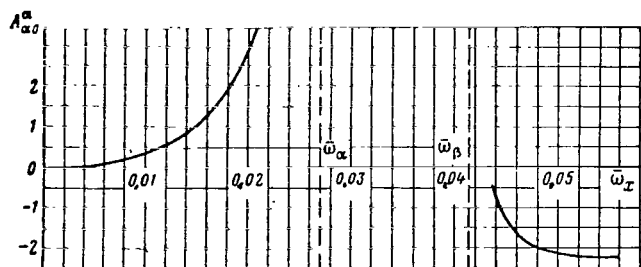
Fig. 4.11.

(In relationships (4.8), (4.9) only the basic terms are retained). Examples of the dependences of the static derivatives $A_{\alpha0}^\alpha$ and $A_{\alpha0}^\beta$ on the size of the angular rolling velocity are given on Figures 4.11 and 4.12. The character of the change in the static derivatives $A_{m\delta_e}^\alpha$, $A_{m\delta_e}^\beta$ are analogous.

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The form of the dependences $\Delta \overline{m}_x(\overline{\omega}_x)$ is determined by whether or not the lateral stability of the aircraft $m_x^\beta(\alpha)$ facilitates development of roll or, on the contrary, inhibits it. Taking into account that usually the derivative $m_x^{\alpha\beta} < 0$ (see Fig. 4.9), from Formula (4.7) is easy to determine the conditions for which the lateral stability facilitates the development of roll of an aircraft. It is obvious that only under such conditions is a system of inertial rotation possible. In order that the lateral stability of the aircraft facilitate the development of roll, it is necessary that the moment from the lateral stability of the aircraft have the same sign as the moment from the ailerons, which does happen when the following relationship is satisfied between the angles of attack α $\underline{/146}$ and side slip β of the aircraft (in a steady system between α_{ss} and β_{ss}):

$$\text{sign } \overline{\omega}_x \cdot (\alpha - \alpha^*)\beta < 0, \tag{4.10}$$

Where $(-\alpha^*)$ is the angle of attack in which $m_x^\beta(-\alpha^*) = 0$ (see Fig. 4.9).
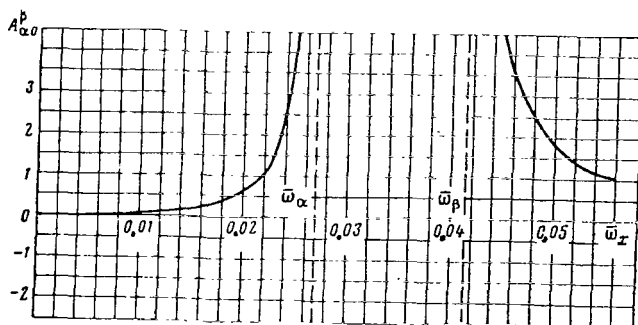
Fig. 4.12.

The relationship between the angles α and β, for which the inequality has the oppo- site sign, corresponds to a deceleration effect of the lateral stability of the air- craft on the angular rolling velocity. From Figures 4.11 and 4.12, it follows that when the value α^* is not very high ($m_{x_0}^\beta \sim 0$), condition (4.10) for the angular rolling velocity near both to the first and second critical rolling velocities, is not satisfied either for such values of the original trim angle of at- tack α_0 (the same is true also for any value of $\Delta m\delta_e$). From this there follows the quite important conclusion that in the range of spatial motions of an aircraft with angular rolling velocity $\omega_x > \overline{\omega}_\beta$ there may be no singular point when $\Delta \overline{m}_x = 0$ and only one stable singular point at the origin when $\omega_x = 0$. This means that regard- less of the position of the elevator during the time of the maneuvers and the size of the angular rolling velocity, placing the ailerons into neutral position will stop the rotation of the aircraft. More- over from Figures 4.11 and 4.12, it follows that for all rolling maneuvers the lateral moment due to the development of angles of attack and side slip located in a given relationship inhibits the development of the angular rolling velocity. As a result the static dependence $\Delta \overline{m}_x(\overline{\omega}_x)$, both for positive and for negative trim angles of attack of the aircraft, at the beginning of the maneuver has for all angular rolling velocities less than the first critical

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a positive derivative $\partial \Delta \bar{m}_x / \partial \bar{\omega}_x$. On Figure 4.13 are shown examples of the functions $\Delta \bar{m}_x(\bar{\omega}_x)$ for different conditions of trim of the aircraft.
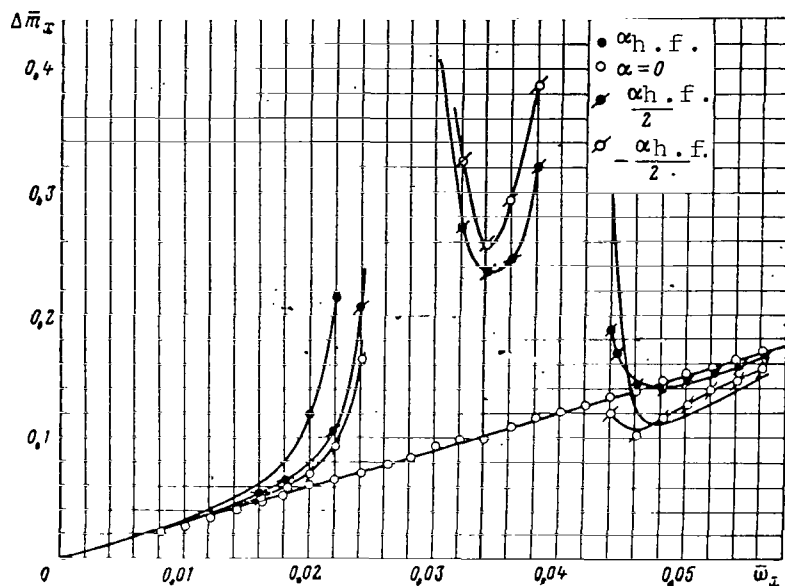
Fig. 4.13

Let us look briefly at the characteristics of the spatial motion of an aircraft with deflection of ailerons carried out from the conditions of horizontal flight (Fig. 4.14). At small angles of deflection of the ailerons, the steady value of the angular rolling velocity does not exceed the value of the first critical, i.e., solutions for the first branch of the static curve $\Delta \bar{m}_x(\bar{\omega}_x)$ is realized. However, at larger deflections of the ailerons, it is possible to realize a solution of the second branch of the static curve which corresponds to rotation at an angular rolling velocity, exceeding the second critical. Computations indicate that placing the ailerons into neutral position stops the rotation of the aircraft (Fig. 4.15). Figure 4.16 shows examples of transient conditions for the basic parameters of motion of an aircraft during entry into roll from the conditions of flight with negative G-force ($\alpha_0 = \alpha_{h.f.}/2$). It is easy to see that the "escape" of the aircraft to large angular rolling velocities, exceeding the second critical, in this case is significantly more simple than during rolling maneuvers with positive original G-forces (explained by the small value of the excess lateral stability of the aircraft at the beginning of the maneuver). In the example under analysis

$$m_x^\beta \left(-\frac{\alpha_{h.f.}}{2} \right) \approx 0.$$

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Fig. 4.14

Fig. 4.15

$\alpha_0 = \alpha_{h.f.}$ $\delta_a = 6°$

$\delta_a = 8,5°$

$\delta_a = 12,5°$

$\alpha_0 = \alpha_h f$ $\delta_a = 18°$ $\delta_r = 0$

$\alpha_0 = -\dfrac{\alpha_{h.f.}}{2}$

$\delta_a = 6°$

$\delta_a = 12,5°$

$\delta_a = 8,5°$
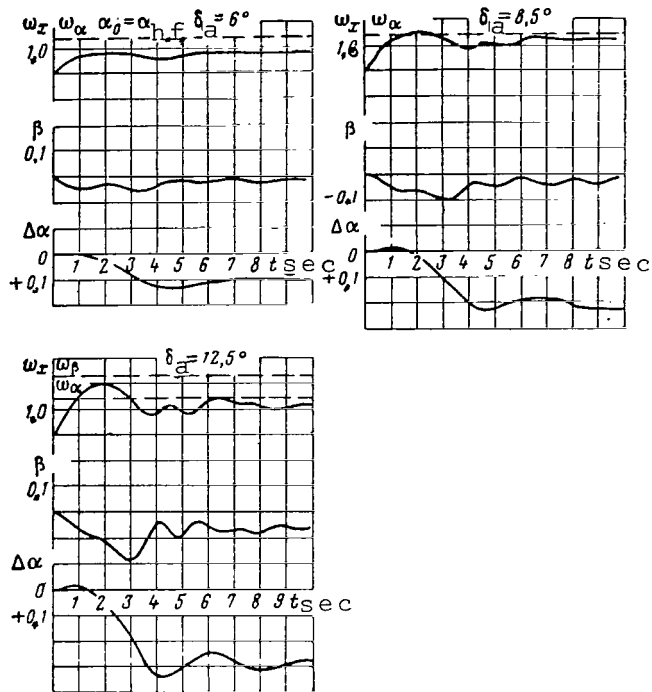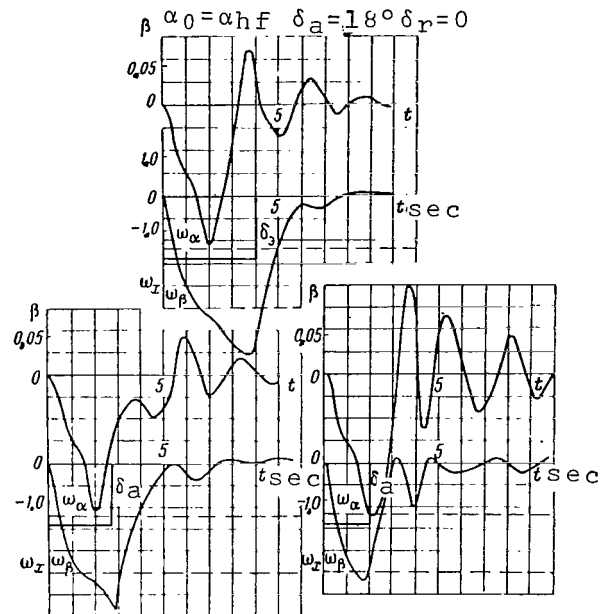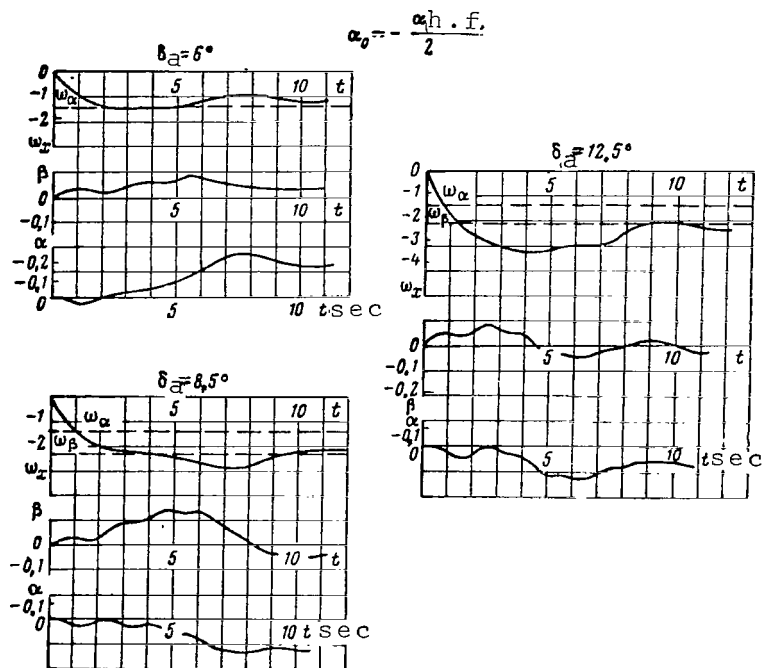
Fig. 4.16

In this case, just as during rolling maneuvers carried out from the conditions of horizontal flight, placing the ailerons into neutral position will stop the rolling rotation of the aircraft (Fig. 4.17).

Characteristics of Control of an Aircraft by the Ailerons when $\overline{\omega}_\alpha \gg \overline{\omega}_\beta$

Relationship of the critical rolling velocities when the inequality $\overline{\omega}_\alpha \gg \overline{\omega}_\beta$ is satisfied is characteristic for the flight of an aircraft at supersonic speed, however under certain conditions it may also be valid at subsonic flying speeds. Typical functions of the derivatives $A^\alpha_{\alpha 0}$ and $A^\beta_{\alpha 0}$ for this case are shown in Figures 4.18 and 4.19. Taking into account condition (4.10), from analysis of the curves given on Figures 4.18 and 4.19, it follows that with an angular rolling velocity exceeding the second critical, the lateral stability of the aircraft during the rolling maneuver facilitates the development of roll at all possible longitudinal trims of the aircraft (both when $\alpha_0 > 0$, and when $\alpha_0 < 0$). (An example of the function $\Delta m_x(\omega_x)$ is shown on Figure 4.20). This leads to the fact that, regardless of the position of the elevator ($\Delta m \delta_e$) during the time of the rolling maneuver for which the angular rolling velocity exceeds the value of the second critical velocity, placing the ailerons into neutral position will not always stop rotation. In some instances it is possible for the rotation of the aircraft to
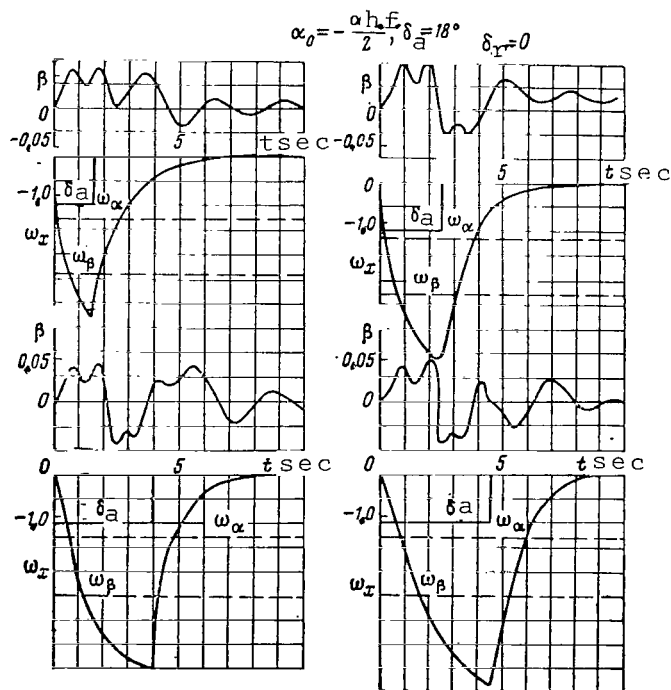
Fig. 4.17
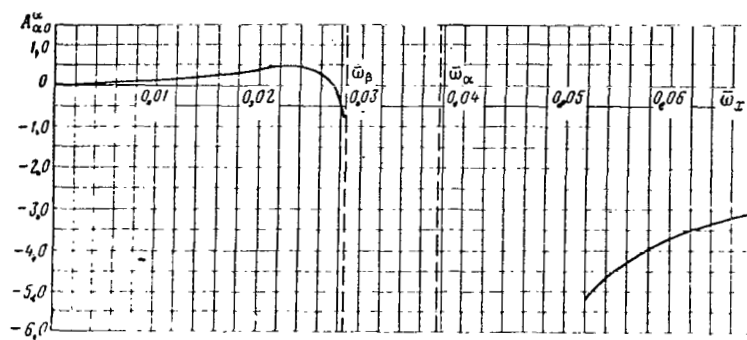
Fig. 4.18.
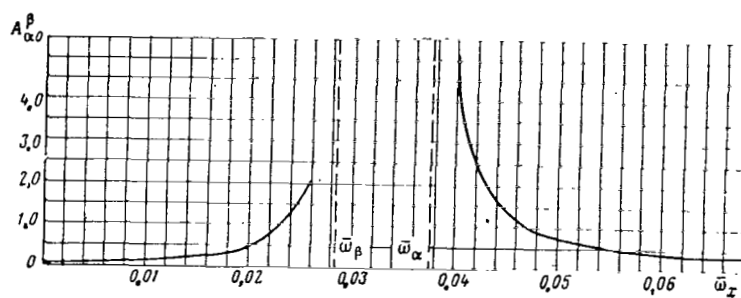
Fig. 4.19
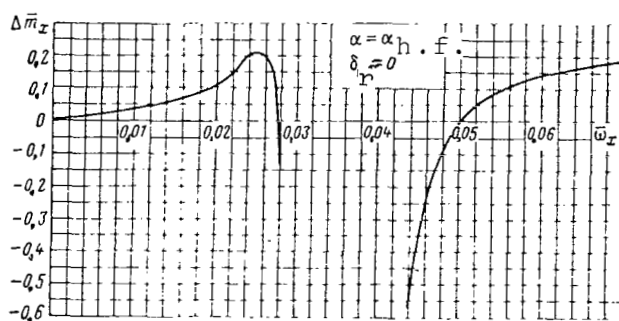
Fig. 4.20.

be maintained during the rolling with the ailerons placed into neutral position (δ_a = 0), usually accompanied by large longitudinal and lateral G-forces. Mathematically this is explained by the presence of a stable singular point when $\Delta m_x = 0$, the angular rolling velocity $\bar{\omega}_x$ in which exceeds the value of $\bar{\omega}_\alpha$.

The form of the transient conditions with gradual deflection of the ailerons is shown on Figure 4.21. From Figure 4.21 it is obvious that the aircraft during rolling maneuvers carried out from conditions of flight with a negative angle of attack ($\alpha_0 = -\alpha_{h.f.}/2$), converts easily to a large angular rolling velocity.

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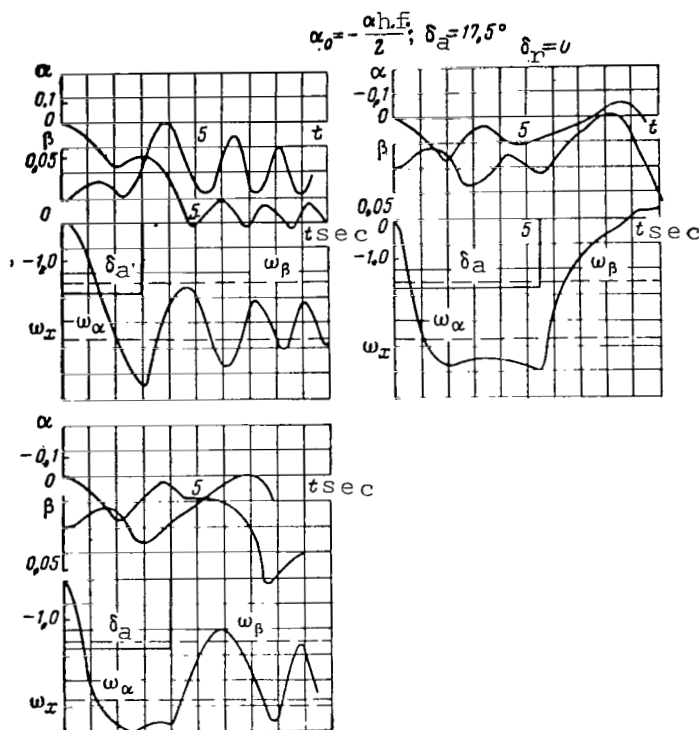
Fig. 4.21

In this case in certain instances placing the ailerons into neutral position will stop the rotation but in some instances it will not. We can note the following law. If at the moment of placing the ailerons into the neutral position, from condition (4.10) it follows that with running values of the angles of attack α and side slip β the lateral stability of the aircraft facilitates development of rolling, placing the ailerons into the neutral position will not stop the rotation. The rolling rotation of the aircraft is maintained due to the lateral stability of the strongly rotating side slip β. It is obvious that the side slip of an aircraft, for retaining the mean value of the angular rolling velocity constant, a

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moment must be created equal to the moment from the ailerons.
Hence it follows that the mean value of the angle of side slip
after placing the ailerons into neutral position will satisfy the
inequality

$$\beta_{\text{av}} \geqslant \frac{-\overline{m}_x^{\overline{\omega}_x} \cdot \overline{\omega}_\alpha}{\overline{m}_x^\beta(\alpha)}. \tag{4.11}$$

In this case in the transient condition we may observe significant
overshoot for the angles β and α. The values β_{ss} and α_{ss} corre-
sponding to the motion with $\omega_x > \overline{\omega}_\alpha$ and $\Delta m_x = 0$, are found with the
help of the curve of the static solutions as β and α for that value
of the angular rolling velocity $\overline{\omega}$ when the static curve $\Delta \overline{m}_x(\overline{\omega}_x)$
intersects the axis $\Delta m_x = 0$.

The motion of an aircraft with $\overline{\omega}_x > \overline{\omega}_\alpha$ ($\overline{\omega}_x$ is greater than the
second critical rolling velocity) is characterized by the reversal
of the sign of coupling between $\Delta\alpha_{ss}$ and $\Delta\overline{m}\delta_e$ (or α_0) to the oppo-
site, with respect to which the motion of the aircraft trimmed
prior to deflection of the ailerons and the beginning of the rolling
maneuver in horizontal flight ($n_{y0} = 1$) is accompanied by the
development of negative angles of attack (negative G-forces). With
the escape of the aircraft into angular rolling velocities exceeding
the second critical, from conditions of flight with negative origi-
nal G-force and the retention of the stabilizer during the entire
time of the maneuver in this trim position, a positive angle of
attack is developed.

20. Deriving the Necessary Conditions for which a System of Inertial Rotation of an Aircraft is Possible

Let us look at a somewhat more general form of the condition
for which a system of inertial rotation of an aircraft is possible
in the case of arbitrary relationships between $m_{\alpha 0}^\beta$ and $m_x^{\alpha\beta}$.

For the existence of a system of inertial rotation of an air-
craft the following are necessary:

(a) The existence of a zero in the function $\Delta m_x(\overline{\omega}_x, \alpha_0, \Delta m_y)$
when $\overline{\omega}_x > \omega_{2crit}$;

(b) A static stability of motion in the vicinity of the
singular point;

(c) A dynamic stability of motion "in the small";

(d) Dynamics of the process of control (speed of readjusting
the controls, etc.); for which the stability of motion is retained
"in the large" in the vicinity of the singular point.

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The first two conditions are necessary but not sufficient conditions for the existence of systems of inertial rotation, i.e., for certain processes of control of an aircraft a system of inertial rotation may also not be realized regardless of the satisfaction of these conditions, for example with slow deflections of the ailerons the system of inertial rotation may be realized but with fast deflection may not be realized, etc. Below we will analyze only the first two conditions (a) and (b) with the additional limitations $\Delta \overline{m}_y = 0$.

In the general case the excess lateral stability of an aircraft $\overline{m}_x(\alpha,\beta)$ is a certain nonlinear function of α and β. For simplification of computation the analysis will be carried out for the case which is of the most practical interest, when the lateral stability $m_x(\alpha,\beta)$ depends on the angle of side slip β and the product $(\alpha \cdot \beta)$.

Let us determine certain general properties of the function of the value of the required moment from the ailerons $\Delta m_x(\overline{\omega}_x)$ which, by taking into account the discussions given above, we can write in the form

$$-\Delta \overline{m}_x = \overline{m}_x^{\overline{\omega}_x} \cdot \overline{\omega}_x + \left[\overline{m}_{x0}^\beta + \overline{m}_x^{\alpha\beta}(\alpha_0 + \alpha_{SS}) \right] \beta_{SS} .\tag{4.12}$$

Let us introduce the function $k\alpha$ with the help of the following relationship:

$$k_\alpha = \frac{\alpha_0 + \alpha_{SS}}{\beta_{SS}}.\tag{4.13}$$

Taking relationship (4.13) into account we find

$$-\frac{\Delta \overline{m}_x}{\overline{\omega}_x} = \overline{m}_x^{\overline{\omega}_x} + \overline{m}_{x0}^\beta \left(\frac{\beta_{SS}}{\overline{\omega}_x}\right) + \overline{\omega}_x \overline{m}_x^{\alpha\beta} \cdot k_\alpha \left(\frac{\beta_{SS}}{\overline{\omega}_x}\right)^2 .\tag{4.14}$$

Introduction of the coupling coefficient between α_{SS} and β_{SS} permits us to obtain the dependence of $\Delta \overline{m}_x$ on one parameter $(\beta_{SS}/\overline{\omega}_x)$.

The condition for the existence of a system of inertial rotation of an aircraft is the presence of a zero in the function $\Delta \overline{m}_x / \overline{\omega}_x$ in the vicinity of the second critical rolling velocity. Let us look first at several specific cases of change in the function $(\Delta \overline{m}_x / \overline{\omega}_x)$, which we shall denote by the letter Y:

$$Y = a_2(\overline{\beta})^2 + a_1 \overline{\beta} + a_0 \quad (a_0 < 0),\tag{4.15}$$

where

$$\bar{\beta} = \frac{\beta_{ss}}{\omega_x}.$$

1. The case when $a_2 \equiv 0\,(m_x^{\alpha\beta} = 0)$. Two types of changes in the function $Y(\beta)$ are possible (Fig. 4.22,a). When $a_1 < 0$, which corresponds to a negative value of $\overline{m}_{x0}^{\beta}$, the funtion $Y(\beta)$ has a zero with a negative value of the quantity (β_{ss}/ω_x). A system of inertial rotation is possible during rolling maneuvers carried out from conditions of flight with a negative G-force ($\alpha_0 < 0$). This case is similar to that studied in Section 17.

2. The case when $a_1 = 0\,(m_{x0}^{\beta} = 0$, or in other definitions $\alpha^{*} = 0$). The function $Y(\beta)$ is symmetric relative to the axis $\beta = 0$ (see Fig. 4.22,b) and has two zeros in that case when $a_2 > 0$. This /155 case is similar to that analyzed above in Section 19, where it was shown that a system of inertial rotation is possible in the original trim of the aircraft, if $\overline{\omega}_{\beta} \ll \omega_{\alpha}$. When $a_2 < 0$, there are no zeros in the function and the system of inertial rotation is impossible.

3. The general case. Let us confine ourselves to an analysis of the function $Y(\bar{\beta})$ for $a_1 < 0$. From expression (4.15) it follows that when $a_2 > 0$, the function $Y(\bar{\beta})$ always has two roots (two zeros) of different signs (see Fig. 4.22,c). In this condition a system of inertial rotation is possible for any original longitudinal trim of the aircraft.

When $a_2 < 0$ the function $Y(\bar{\beta})$ may in general not have real roots, or may have two negative roots. The condition for the existence of even one real root of the function $Y(\bar{\beta})$ is the requirement for a positive discriminant which is written in the form
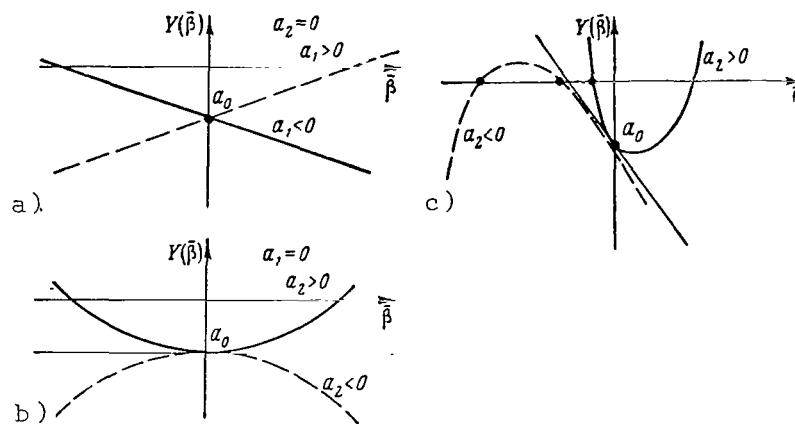
Fig. 4.22

$$a_1^2 \geqslant 4a_0 a_2. \tag{4.16}$$

If we substitute into expression (4.16) the values of the aerodynamic characteristics of an aircraft, we find the necessary but insufficient conditions for the existence of systems of inertial rotation in the general case.

$$\frac{\left(m_{x0}^\beta\right)^2}{\overline{\omega}_x^2} \geqslant 4\overline{m}_x^{\omega_x} \cdot \overline{m}_x^{\alpha\beta} \cdot k_\alpha. \tag{4.17}$$

When $(\overline{m}_x^{\alpha\beta} \, k\alpha) > 0$, inequality (4.17) is always satisfied. Let us note that from the two zeros of the function $Y(\beta)$ (see Fig. 4.22,c) one corresponds to the statically unstable solution. The graph showing the changes in the function $\Delta\overline{m}_x(\overline{\omega}_x)$ is shown on Figure 4.23.

Let us look at the use of the necessary condition for the /156 existence of a system of inertial rotation in specific cases for various relationships of the critical angular rolling velocities $\overline{\omega}_\alpha$, $\overline{\omega}_\beta$ on which depends the form of the function $k\alpha$ in the vicinity of the second critical rolling velocity.

$$(1) \qquad \overline{\omega}_\alpha \gg \overline{\omega}_\beta.$$

Fig. 4.23

Let us determine the quantity k_α when $\overline{\omega}_x \simeq \overline{\omega}_\alpha$:

$$k_\alpha = \frac{\dfrac{-A_0}{\mu^2\overline{\omega}_x^2} + \left(\overline{m}_y^\beta + B\mu\overline{\omega}_x^2\right)A\mu + \overline{m}_y^{\omega_y} \cdot \overline{m}_z^{\omega_z}{}_{\mathrm{b}}}{\overline{m}_{z\mathrm{b}}^\alpha \, \overline{m}_y^{\omega_y} + A\mu\overline{\omega}_x^2 B \, \dfrac{c_y^\alpha}{2} + \dfrac{c_y^\alpha}{2\mu} \, \overline{m}_{z\mathrm{b}}^{\omega_z} \overline{m}_y^{\omega_y}}. \tag{4.18}$$
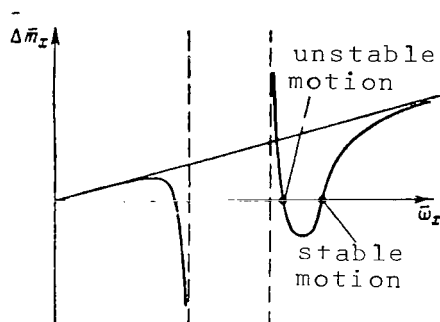
Taking into account that $\overline{\omega}_x \to \overline{\omega}_\alpha$ and the approximate relationships are satisfied for the critical rolling velocities

$$\overline{\omega}_\alpha^2 = -\frac{\overline{m}_{z\mathrm{b}}^\alpha}{A\mu} \; ; \; \overline{\omega}_\beta^2 = \frac{-\overline{m}_y^\beta}{B\mu}, \tag{4.19}$$

we find

$$k_\alpha \cong -\frac{\left[B\mu\left(\overline{\omega}_\alpha^2 - \overline{\omega}_\beta^2\right) + \dfrac{\overline{m}_y^{\omega_y} \, \overline{m}_{z\mathrm{b}}^{\omega_z}}{A\mu}\right]}{\overline{\omega}_\alpha^2\left(B \, \dfrac{c_y^\alpha}{2} - \overline{m}_y^{\omega_y}\right)}. \tag{4.20}$$

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From expression (4.20) it is obvious that the function k_α is always negative and consequently when $\overline{m}_x^{\alpha\beta} < 0$ the system of inertial rotation is always possible and there are two values of $\overline{\beta}$ corresponding to this system. When $\overline{\omega}_\alpha \gg \overline{\omega}_\beta$, expression (4.20) can be simplified.

$$k_\alpha = -\frac{B\mu}{\left(B\dfrac{c_y^\alpha}{2} - \overline{m}_y^{\overline{\omega}y}\right)}\left[1 - \left(\frac{\overline{\omega}_\beta}{\overline{\omega}_\alpha}\right)^2\right].$$

In the case when $m_x^{\alpha\beta} > 0$, the system of inertial rotation is possible in satisfying conditions (4.17) which may be rewritten in the form

$$\frac{\overline{m}_x^{\alpha\beta}}{(\overline{m}_{x0}^\beta)^2} \leqslant \frac{\left(B\dfrac{c_y^\alpha}{2} - \overline{m}_y^{\overline{\omega}y}\right)\cdot A}{4\overline{m}_x^{\overline{\omega}x}\left[1 - \left(\dfrac{\overline{\omega}_\beta}{\overline{\omega}_\alpha}\right)^2\right]\overline{m}_{z\mathrm{b}}^\alpha B}.$$

(4.22)

An example of the boundary of the region of the parameters $\overline{m}_x^{\alpha\beta}$ and m_{x0}^β, for which a system of inertial rotation is possible, is shown on Figure 4.24.

(2) $\overline{\omega}_\alpha \ll \overline{\omega}_\beta$.
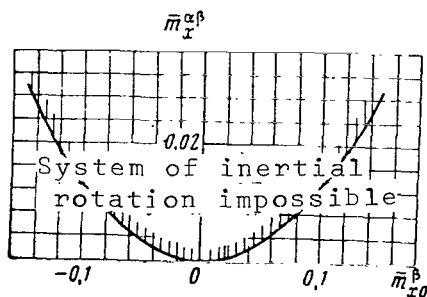
From Formula (4.18) it follows that with such a relationship of the critical rolling velocities the value of the coefficient k_α is quite sensitive to selection of the value $\overline{\omega}_x$. We can show that in the range of angular velocities greater than the second critical, the quantity k_α is contained in certain limits.

Fig. 4.24.

$$k_{\alpha\max} > k_\alpha > k_{\alpha\min},$$ (4.23)

where $k_{\alpha\max} > 0$ but $k_{\alpha\min} < 0$. Returning to Formula (4.17) we find that for $\overline{m}_x^{\alpha\beta} < 0$ the system of inertial rotation is possible, but /158 as a function of the quantity k_α its probability is different. We can assume that the closer the angular rolling velocity is to the critical, the more probable will be the development of a system of inertial rotation.

Let us determine the expression $k_{\alpha\max}$. The angular rolling velocity for which $k_\alpha = k_{\alpha\max}$ is determined from the condition $A_0(\overline{\omega}_x) = 0$, whence follows

$$k_{\alpha\,max} = \frac{\left[\dfrac{\overline{m}{}^{\alpha}_{z}b}{\overline{m}{}^{\alpha}_{z}b + A\mu\overline{\omega}{}^{2}_{x}} - 1\right]\left(-A\dfrac{c^{\beta}_{z}}{2} - \overline{m}{}^{\overline{\omega}}_{z}b\right)\left(\dfrac{c^{\alpha}_{y}}{2}B - \overline{m}{}^{\overline{\omega}}_{y}\right) + \overline{m}{}^{\overline{\omega}}_{y}\cdot\overline{m}{}^{\overline{\omega}}_{z}b}{\overline{m}{}^{\alpha}_{z}b\;\overline{m}{}^{\overline{\omega}}_{y} + A\mu\overline{\omega}{}^{2}_{x}B\dfrac{c^{\alpha}_{y}}{2}}. \qquad (4.24)$$

Using condition (4.17) and expression (4.24) we can plot the regions of the parameters ($\overline{m}{}^{\alpha\beta}_{x}$, $\overline{m}{}^{\beta}_{x\,0}$) with a different degree of probability of the system of inertial rotation analogous to that shown on Figure 4.24. The results obtained are given in Table 6.

TABLE 6

$\omega_{\alpha} \gg \omega_{\beta}$ (Characteristic for $M \gg 1$)	
$m^{\alpha\beta}_{x} < 0$	$m^{\alpha\beta}_{x} > 0$
A system of inertial rotation is possible when	A system of inertial rotation is possible if the following inequality is satisfied
$\alpha_{0} > 0$ and $\alpha_{0} < 0$	$\overline{m}{}^{\alpha\beta}_{x} \leqslant \dfrac{(m^{\beta}_{x0})^{2}\left(B\dfrac{c^{\alpha}_{y}}{2} - \overline{m}{}^{\overline{\omega}}_{y}\right)A}{2\overline{m}{}^{\overline{\omega}}_{x}\overline{m}{}^{\alpha}_{z}\left(1 - \dfrac{\overline{\omega}{}^{2}_{\beta}}{\overline{\omega}{}^{2}_{\alpha}}\right)B}$
	(for $\alpha_{0} < 0$)
$\omega_{\alpha} \ll \omega_{\beta}$ (Characteristic for $M \ll 1$)	
$m^{\alpha\beta}_{x} < 0$	$m^{\alpha\beta}_{x} > 0$
A system of inertial rotation is possible when the following inequality is satisfied	A system of inertial rotation is possible when $\alpha_{0} > 0$ and $\alpha_{0} < 0$.
$(m^{\beta}_{x0})^{2} \geqslant 4\overline{m}{}^{\overline{\omega}}_{x}\overline{\omega}{}^{2}_{\beta}\overline{m}{}^{\alpha\beta}_{x}k_{\alpha}$	

21. Approximate Analytical Evaluations of the Maximal Angle of Side Slip During Maneuvers Accompanied by Strong Rolling.

In piloting an aircraft, maneuvers accompanied by rolling are /159 basic. In connection with this it is quite important that we look at the problem of determining G-forces which act on the aircraft in carrying out such maneuvers. The flying tests and computations show that G-forces which occur during flying maneuvers, both normal and lateral, may reach significant values. In particular, according to the published data, it is known that during rolling maneuvers of the American experimental aircraft X-2 there were catastrophes associated with breakdown in the vertical tail group under the influence of large lateral G-forces [37], [38], [51], [59].

In this section we have summarized several results obtained in studying the dynamics of an aircraft and the analysis of the G-forces acting on it, with the simultaneous control of ailerons and elevators. With respect to the large number of parameters which influence the dynamics of controlled motion of an aircraft the results given below are not sufficiently complete. The investigations which encompass all questions of the dynamics of controlled flight of an aircraft with analysis of the maximal G-forces apparently can be carried out only for a specific aircraft design.

We will look only at the qualitative characteristics of the transient conditions of an aircraft during rolling maneuvers, since to obtain quantitative results we must carry out a more precise analysis of the dynamics taking into account the nonlinearity of the dynamic characteristics of the aircraft, for example, such as

$$m_y^\beta(\alpha), \quad m_z^z(\alpha), \quad m_x^{\delta a}(\alpha) \ etc.$$

Let us introduce approximate formulas for the characteristics of the transient conditions based on the basic parameters of motion of an aircraft during rolling maneuvers carried out with the help of the ailerons from conditions of flight with a positive normal G-force n_{y0}. Since the solutions to the equations of motion for an arbitrary law of deflection of ailerons in time cannot be obtained in analytical form, to find the approximate qualitative results we must narrow the class of controlled processes to be studied taking as a basis certain rather simple model cases. As such a rolling maneuver let us look at the motion of an aircraft after deflection of the ailerons. Such a maneuver may be valid in carrying out advanced flying patterns called "roll", and also corresponds in the general case to the beginning of a rolling maneuver of an aircraft. We shall assume that the initial conditions are the conditions of flight with a constant positive G-force n_{y0} (i.e., $\alpha_0 > 0$) (in the specific case of horizontal flight). Let us look at the case when the critical angular rolling velocity corresponding to a yawing motion is smaller than the critical velocity i.e., $\bar{\omega}_\beta \ll \bar{\omega}_\alpha$, which /160

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is characteristic of supersonic flying speeds. With such relation-
ships of the critical rolling velocities the greatest interest is
that of analyzing the change in the lateral G-force or the angle of
side slip of the aircraft as a function of time. As was found in
analyzing the curves of the static solutions and as a result of the
numerical computations of the transient conditions, the gradual
deflection of the ailerons leads to motion of the aircraft with an
angular rolling velocity accompanied by intense development of
side slip. Change in the angle of side slip has an oscillational
character, sometimes with large overshoot, which may be dangerous
from the point of view of stability of the aircraft and the effect
of lateral G-forces on the pilot.

Let us find approximate analytical evaluations for solutions
to the system of equations of motion by looking at the quantities
ω_x and α_0 as known functions of time determined by the deflection
of the ailerons and elevator, respectively. We can determine them
by using the value of the maximal angle of side slip β_{max} during
the rolling maneuver. The simplified assumption concerning the
fact that ω_x is a known function of time will limit application of
the results obtained for aircraft with a small value of lateral
stability ($m_x^\beta \sim 0$).

The course of further discussions includes the following. The
equations of longitudinal motion of an aircraft are simplified in
such a way that the order of the system of equations of motion of
an aircraft is reduced to the second. The equations obtained with
variable coefficients are approximately integrated and the formula
for β_{max} is derived.

Let us look at the equations of longitudinal motion of an air-
craft during a rolling maneuver (1.33):

$$\left.\begin{array}{l} \dfrac{d\alpha}{d\tau} + \dfrac{c_y^\alpha}{2}\alpha - \mu\bar{\omega}_z = -\mu\beta\bar{\omega}_x; \\[2mm] \dfrac{d\bar{\omega}_z}{d\tau} - \overline{m_{z\mathrm{b}}^{\bar{\omega}_z}}\cdot\bar{\omega}_z - \overline{m_{z\mathrm{b}}^\alpha}\alpha = -A\mu\bar{\omega}_x\bar{\omega}_y. \end{array}\right\}$$

(4.25)

With the help of simple computations we can transform these equa-
tions to the form

$$\left.\begin{array}{l} \dfrac{d^2\alpha}{d\tau^2} + \left(-\overline{m_{z\mathrm{b}}^{\bar{\omega}_z}} + \dfrac{c_y^\alpha}{2}\right)\dfrac{d\alpha}{d\tau} + \mu\left(-\overline{m_{z\mathrm{b}}^\alpha} - \dfrac{c_y^\alpha \overline{m_{z\mathrm{b}}^{\bar{\omega}_z}}}{2\mu}\right)\alpha = \\[3mm] = A\mu\bar{\omega}_x\bar{\omega}_y - \mu\bar{\omega}_x\dfrac{d\beta}{d\tau} + \overline{m_{z\mathrm{b}}^{\bar{\omega}_z}}\mu\beta\bar{\omega}_x - \mu\beta\dfrac{d\bar{\omega}_x}{d\tau}; \\[4mm] \dfrac{d^2\bar{\omega}_z}{d\tau^2} + \left(-\overline{m_{z\mathrm{b}}^{\bar{\omega}_z}} + \dfrac{c_y^\alpha}{2}\right)\dfrac{d\bar{\omega}_z}{dt} + \mu\left(-\overline{m_{z\mathrm{b}}^\alpha} - \dfrac{c_y^\alpha \overline{m_{z\mathrm{b}}^{\bar{\omega}_z}}}{2\mu}\right)\bar{\omega}_z = \\[3mm] = -\overline{m_{z\mathrm{b}}^\alpha}\beta\bar{\omega}_x\mu^2 - A\mu\bar{\omega}_x\dfrac{d\bar{\omega}_y}{d\tau} - A\mu\bar{\omega}_y\dfrac{d\bar{\omega}_x}{d\tau} - A\dfrac{c_y^\alpha}{2}\mu\bar{\omega}_x\bar{\omega}_y. \end{array}\right\}$$

(4.26)

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We try to find extremely simple final formulas and make a
series of assumptions relative to the properties of the solutions
of the system of Equations (4.26). We assume that the natural
frequencies of the aircraft oscillation in angle of attack α and
angular pitching velocity are considerably greater than the fre-
quencies of change of the variables in the right hand sides of
these equations determined by the values $\beta(\tau)$, $\bar\omega_y(\tau)$, $\bar\omega_x(\tau)$ and the
degree of damping of the processes for α and $\bar\omega_z$ are also rather
high. This assumption is valid in those cases when the inequality
$\omega_\alpha \gg \omega_\beta$ is satisfied. With such assumptions the approximate solu-
tion for α and $\bar\omega_z$ can be assumed as quasistatic and written in the
form

$$\alpha \simeq \frac{1}{a_0}\left[-A\mu\bar\omega_x\bar\omega_y - \bar\omega_x\mu\frac{d\beta}{d\tau} + \bar m_{z\mathrm{b}}^{\bar\omega_z}\mu^3\bar\omega_x - \beta\mu\frac{d\bar\omega_x}{d\tau}\right];$$

$$\bar\omega_z \simeq \frac{1}{a_0}\left[-\mu^2\bar m_{z\mathrm{b}}^{\alpha}\beta\bar\omega_x - A\mu\bar\omega_x\frac{d\bar\omega_y}{d\tau} - A\mu\bar\omega_y\frac{d\bar\omega_x}{d\tau} - A\mu\frac{c_y^{\alpha}}{2}\bar\omega_x\bar\omega_y\right],$$

(4.27)

where

$$a_0 = \left(-\bar m_{z\mathrm{b}}^{\alpha} - \frac{c_y^{\alpha}\bar m_{z\mathrm{b}}^{\bar\omega_z}}{2\mu}\right)\mu.$$

(4.28)

Simplified assumptions have permitted lowering the order of
the system of equations of motion of the aircraft from fourth to
second. If we substitute expressions (4.27) and (4.28) into the
equation for β' and $\bar\omega_y'$ and make the necessary transformations we
find the approximate equation of second order for determining
size of the angle of side slip $\beta(\tau)$ in the form

$$\frac{d^2\beta}{d\tau^2} + p(\bar\omega_x)\frac{d\beta}{d\tau} + q(\bar\omega_x)\beta = b_1(\bar\omega_x)\frac{d\bar\omega_x}{d\tau} + b_0(\bar\omega_x)\bar\omega_x,$$

(4.29)

where the functions $p(\bar\omega_x)$, $q(\bar\omega_x)$, $b_1(\bar\omega_x)$ $b_0(\bar\omega_x)$ are found according
to approximate formulas (in the expressions for p, q, b_1, b_0 certain
terms of the type $\bar\omega_x\frac{d\bar\omega_x}{d\tau}$, $\left(\frac{d\bar\omega_x}{d\tau}\right)^2$, etc., have been omitted where their
influence is not substantial):

$$p(\bar\omega_x) = \frac{a_0\left(-\frac{c_z^{\beta}}{2} - \bar m_y^{\bar\omega_y}\right) + AB\bar\omega_x^2\mu^2\left(\frac{c_y^{\alpha}}{2} - \frac{c_z^{\beta}}{2}\right) + \mu^2\bar\omega_x^2\left(-\bar m_y^{\bar\omega_y} - \bar m_{z\mathrm{b}}^{\bar\omega_z}\right)}{a_0 + \bar\omega_x^2\mu^2(1 + AB) + \frac{AB}{a_0}\mu^4\bar\omega_x^4} +$$

$$+ \frac{\frac{AB}{a_0}\mu^4\bar\omega_x^4\left(\frac{c_y^{\alpha}}{2} - A\bar m_{z\mathrm{b}}^{\bar\omega_z}\right)}{a_0 + \bar\omega_x^2\mu^2(1 + AB) + \frac{AB}{a_0}\mu^4\bar\omega_x^4};$$

(4.30)

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$$q(\bar{\omega}_x) = A_0(\bar{\omega}_x)\left[\cfrac{1}{a_0 + \bar{\omega}_x^2\,\mu^2(1+AB) + \cfrac{AB}{a_0}\mu^4\bar{\omega}_x^4}\right];$$

$$b_1(\bar{\omega}_x) = \cfrac{a_0 a_0}{a_0 + \bar{\omega}_x^2\,\mu^2(1+AB) + \cfrac{AB}{a_0}\mu^4\bar{\omega}_x^4};\qquad (4.32)$$

$$b_0(\bar{\omega}_x) = \cfrac{a_0\left(-a_0\overline{m_y^\omega}{}^y + AB\,\cfrac{c_y^\alpha}{2}\mu^2\cdot\bar{\omega}_x^2\right) + \bar{\omega}_{x0}B(a_0 - A\mu^2\bar{\omega}_x^2)}{a_0 + \bar{\omega}_x^2\,\mu^2(1+AB) + \cfrac{AB}{a_0}\mu^4\bar{\omega}_x^4}.$$

(4.33)

Let us note that all the functions (4.30)-(4.33) are even functions of the angular rolling velocity ω_x. The graphic character of their change is shown on Figures 4.25 and 4.26.
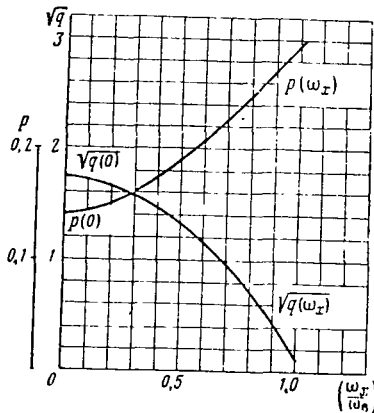
Fig. 4.25
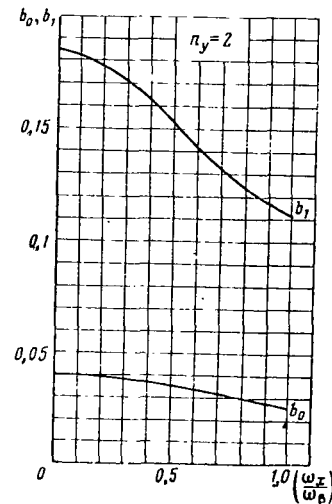
Fig. 4.26

After all the simplifications made for finding the rules of time change in the angle of side slip $\beta(\tau)$ we find the linear non-uniform differential equation of second order with variable coefficients (since the quantity $\bar{\omega}_x$ is a known function of time). Let us find an approximate solution to the equation for the gradual deflection of the ailerons with zero initial conditions for all parameters of motion. We shall look at the equation for the angular rolling velocity of an aircraft ω_x in the most simple form by as-   /163
suming that the aircraft possesses a negligible small lateral stability.

$$\frac{d\overline{\omega}_x}{d\tau} - \overline{m}_x^{\overline{\omega}_x}\overline{\omega}_x = \overline{m}_x^{\delta_a}\delta_a \qquad (4.34)$$

The solution for $\overline{\omega}_x(\tau)$ for a gradual deflection of ailerons and zero initial conditions is determined from the formula

$$\overline{\omega}_x = \Omega(1 - e^{-\lambda_0\tau}), \qquad (4.35)$$

where

$$\Omega = -\frac{\overline{m}_x^{\delta_a} \cdot \delta_a}{\overline{m}_x^{\overline{\omega}_x}} ;$$

$$\left.\begin{array}{c}\\ \\ \\ \\ \end{array}\right\}$$

$$\lambda_0 = -\overline{m}_x^{\overline{\omega}_x}. \qquad (4.36)$$

Let us seek the solution for the angle of side slip in the form

$$\beta(\tau) = \beta_{un}(\tau) + \beta_{sp}(\tau), \qquad (4.37)$$

where β_{un} is the approximate general solution for the uniform equation; and β_{sp} is the approximate specific solution for the non-uniform Equation (4.29).

To find β_{un} and β_{sp} we use known approximate asymptotic relationships [7], [61], which permit finding the solution to the linear equation with variable coefficients and in first approximation we find

$$\beta_{un} \simeq \frac{B_0}{\sqrt[4]{q(\tau)}} e^{-\frac{1}{2}\int_0^\tau p(\tau)d\tau} \cdot \sin\left(\int_0^\tau \sqrt{q(\tau)}\,d\tau + \varphi_0\right); \qquad (4.38)$$

$$\beta_{sp} \simeq \frac{b_1\dfrac{d\overline{\omega}_x}{d\tau} + b_0\overline{\omega}_x(\tau)}{q(\tau)} . \qquad (4.39)$$

The arbitrary constants B_0 and ϕ_0 can be determined by taking the initial conditions into account

$$\beta(0) = \frac{d\beta}{d\tau}(0) = 0. \qquad (4.40)$$

We find

$$B_0 = -\frac{b_1(0)\,\Omega\lambda_0}{\sqrt[4]{[q(0)]^3}\sin\varphi_0} ; \qquad (4.41)$$

$$\varphi_0 = \arctan \frac{\sqrt{q(0)}}{\frac{b_0(0)}{b_1(0)} + \left[\frac{p(0)}{2} - \frac{\lambda_0}{4}\left(1 + 3\frac{q(\Omega)}{q(0)}\right)\right]}. \qquad (4.42)$$

Numerical calculations show that the value of ϕ_0 is near $-\pi/2$, but $\sin \phi_0 \simeq -1$ (Fig. 4.27). Hence if we assume that the equation $\phi_0 = -\pi/2$ is satisfied, we can reduce the solution for $\beta(\tau)$ to the form

$$\beta(\tau) = \frac{b_1(\tau)\Omega\lambda_0 e^{-\lambda_0\tau} + b_0(\tau)\Omega(1 - e^{-\lambda_0\tau})}{q(\tau)} -$$

$$-\frac{b_1(0)\Omega\lambda_0}{\sqrt[4]{q(0)^3}\sqrt[4]{q(\tau)}} e^{-\frac{1}{2}\int_0^\tau p(\tau)\,d\tau} \cos\left(\int_0^\tau \sqrt{q(\tau)}\,d\tau\right). \qquad (4.43)$$
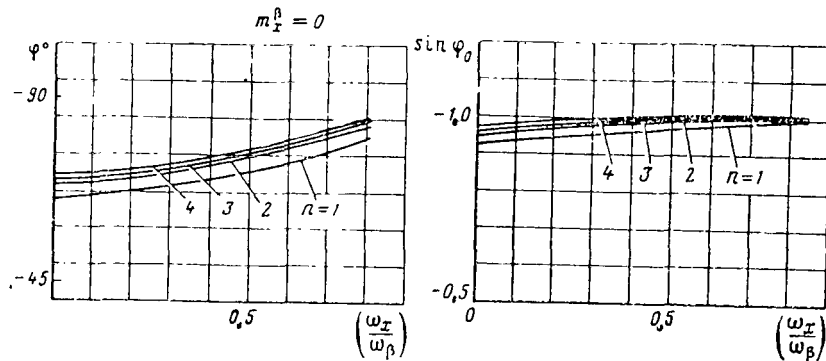
Fig. 4.27

Figure 4.28 shows the results of comparing the solution of Equation (4.43) with the precise one obtained by modeling the equations of motion (1.28)(the approximate solution is given by the dotted line). As is obvious from these figures the agreement of the solutions is totally satisfactory.

Of basic interest is Formula (4.43) for finding the values of the maximal angle of side slip during a rolling maneuver (β_{max}). We may approximately assume that the angle of side slip takes its own maximal value at the moment of time when

$$\int_0^{\tau_s} \sqrt{q(\tau)}\,d\tau = \pi. \qquad (4.44)$$

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Relationship (4.44) is obtained from the condition of equating

$\cos \int_0^\tau \sqrt{q(\tau)} d(\tau)$ to unity with a minus sign which, in the case of a

small damping of motion of an aircraft in pitching and yawing, approximately corresponds to the moment of time that the maximum is obtained for the angle of side slip. We shall also assume that the angular rolling velocity $\omega_x(\tau)$ practically reaches its own steady value $\omega_x(\infty) = \Omega$ much earlier than condition (4.44), i.e., before β reaches its own maximal value.

$$\tau_1 > \frac{3}{\lambda_0}.$$

(4.45)
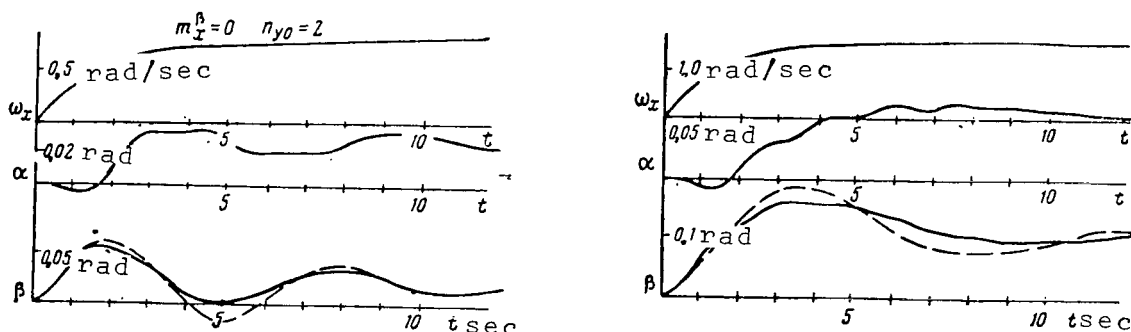
Fig. 4.28.

Under such assumptions, Equation (4.44) may be approximately written in the form

$$\tau_1 = \frac{\pi}{\sqrt{q(\Omega)}}.$$

(4.46)

Taking relationships (4.45) and (4.46) into account, from solution (4.43) we find the approximate expression for the quantity β_{max}:

$$\beta_{max} \simeq \beta_{ss}(\Omega) \left[\sqrt[4]{\left(\frac{q(\Omega)}{q(0)}\right)^3 \frac{b_1(0)}{t_0(\Omega)}} \, e^{-\frac{p(\Omega)}{2} \frac{\pi}{\sqrt{q(\Omega)}}} + 1 \right].$$

(4.47)

The function b_0 depends very little on the quantity Ω (see Fig. 4.27), therefore if we substitute $b_0(\Omega)$ for $b_0(0)$ and express the values of all the variables through the aerodynamic and inertial characteristics of the aircraft, we obtain the final approximate formula for finding the ratio (β_{max}/β_{ss}):

$$\left(\frac{\beta_{max}}{\beta_{ss}}\right) \simeq \sqrt[4]{\overline{\left(\frac{q(\Omega)}{q(0)}\right)^3}} \cdot \frac{\overline{m}_x^{\overline{\omega}_x}}{\overline{m}_y^{\overline{\omega}_y} - B\frac{c_\mu^\alpha}{2}\overline{\frac{n_{y0}-1}{n_{y0}}}} e^{-\frac{p\pi}{2\sqrt{\overline{q}}}} + 1. \qquad (4.48)$$
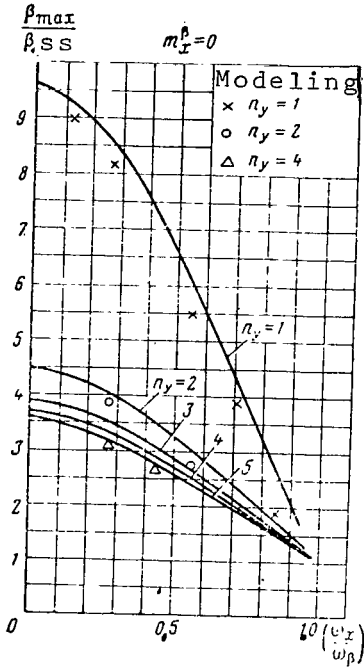
Fig. 4.29
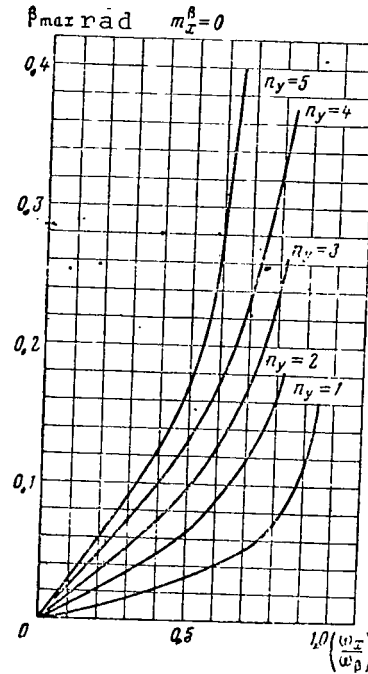
Fig. 4.30

As an example, on Figures 4.29 and 4.30, we have plotted graphs of the dependences (β_{max}/β_{ss}) and β_{max} on Ω, computed using Formula (4.48); using the same points we have plotted the solutions obtained during the modeling. From these figures it follows in particular that the ratio of β_{max} to β_{ss} is decreased with an increase in the size of Ω. This is due in first order to the decrease in the frequency of oscillations and consequently to the increase in the time it takes to reach the maximum for the angle of side slip β. With an increase in the time of the transient condition, the greater part of the energy is dissipated and the amplitude for the angle of side slip β is respectively decreased. Increase in the initial normal G-force n_{y0} during the rolling maneuver also leads to a decrease in the overshoot for the angle of side slip β. However it follows to note that, regardless of the decrease in the /167 relative quantity (β_{max}/β_{ss}), the value β_{max} itself [with increase in the size of the angular rolling velocity with which the maneuver (Ω) is carried out and in the G-force (n_{y0})] is increased since β_{ss} changes more strongly than does the ratio β_{max}/β_{ss} (see Fig. 4.30).

Deriving approximate Formula (4.48) was based on the assumption that $|\overline{m}_{zb}^{\alpha}| \gg |\overline{m}_{y}^{\beta}|$. The graph of the results of proving the effect of this assumption on the results of computation is shown in Figure 4.31 (on the figure we note the solutions obtained during modeling). From Figure 4.31 it follows that the satisfactory agreement of the approximate computations with modeling is valid for the relationship $\overline{m}_{zb}^{\alpha}/\overline{m}_{y}^{\beta} \geqslant 2.0$ ($A \cong B$).

The results obtained above belong to the case when the aircraft does not possess lateral stability ($m_{x}^{\beta} = 0$). As the computations and modeling show, the presence of lateral stability somewhat lowers the size of the overshoot for the angle of side slip of an aircraft during rolling maneuvers. With overshoot for β the size of damping of the aircraft for yawing $\overline{m}_{y}^{\overline{\omega}y}$ also has an effect. For illustration of the qualitative picture of the influence of these parameters, on Figures 4.32 and 4.33 graphs are plotted for the dependences of β_{max}/β_{ss} as a function of Ω.

Fig. 4.31

Fig. 4.32.

Fig. 4.33

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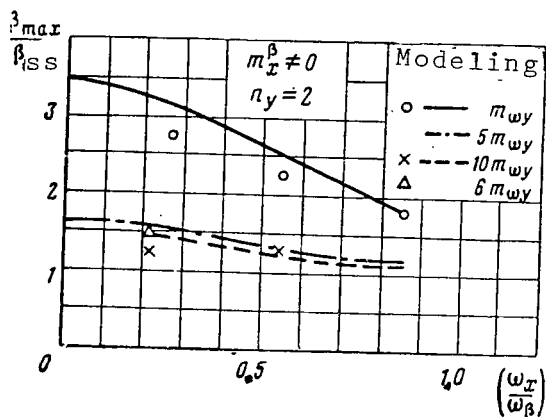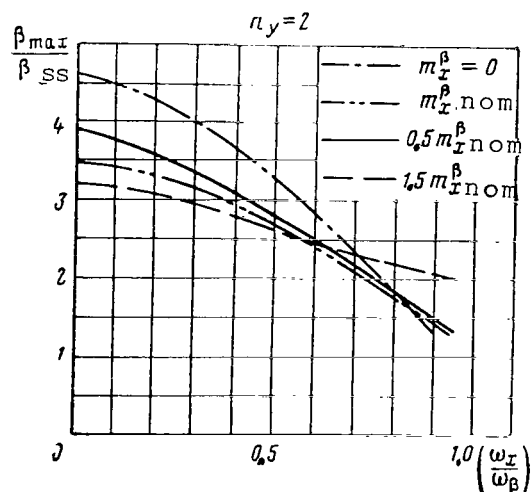
For the original values of the parameter $\overline{m_y^{\omega_y}}$ and m_{x0}^{β} we have taken
several mean values which are characteristic of maneuvering aircraft
with swept wings (with an angle of sweep $\chi \cong 45°$).

Dynamics of an Aircraft in Carrying Out a Turn at a Given Angle of Bank.

Pinsker [50] has analyzed this specific but rather widespread
turning maneuver at a given angle of bank in considerable detail
and systematically.

In this paper, using the systematized modeling, the transient
conditions of an aircraft are studied for the angles of attack and
side slip with the rolling maneuver indicated above, and functions
are plotted for the maximal overshoot for these variables on
several basic parameters which determine the dynamics of the air-
craft.
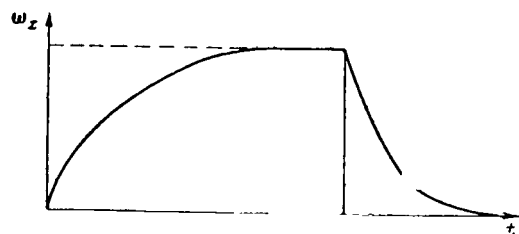
Fig. 4.34

In this paper are studied the
turning maneuver of an aircraft
at a given angle of bank γ, which
is accomplished from the conditions
of horizontal flight with an angle
of attack α_0. In this case it is
assumed that the change in the
angle of rolling velocity is a
known function of time, a typical
graph of which is shown on Figure
4.34. It is assumed that the
angular rolling velocity changes exponentially and in such case the
signs of the exponents for increase and decrease in the angular
velocity are identical. The basic parameters which determine the
motion of the aircraft during rolling maneuvers are the excess
longitudinal and directional stability, with respect to which all
dependences of the size of overshoot for the angles of attack and
side slip are plotted in this paper as functions of these two
parameters. We also study the effect of damping and the ratio of
inertial moments J_y/J_z for the value of the dynamic scattering. In
the paper are contained detailed graphs which permit approximately
determining from the known parameters of the aircraft the maximal
values of the angles of attack and side slip of the aircraft for
such rolling maneuvers.

The basic assumption which limits the range of the problems,
for solution of which the results obtained in [50] are used, is
the form of the transient condition for the angular rolling veloc-
ity (see Fig. 4.34). In the paper it is noted that such a charac-
ter of the transient condition for ω_x may be guaranteed by the pi-
lot using the manual control. Such a conclusion is correct in the
entire range of angular rolling velocities for aircraft in which
the value of the lateral stability $\overline{m_x^{\beta}}$ is negligibly small.

For aircraft possessing a rather large degree of lateral stability, the results obtained in [50] are applicable only in the ranges of stable motion with angular rolling velocity ω_x less than the first critical. In fact, as was shown above, the seeming effectiveness of the ailerons in carrying out maneuvers from conditions of flight with a positive angle of attack is limited, thus leading in several instances to the impossibility of obtaining steady angular rolling velocities exceeding the first critical. Analogously, for maneuvers with original negative angles of attack, the motion of the aircraft is unstable in the range between the first and second critical angular rolling velocities and apparently it is impossible to guarantee transient conditions of the type shown in Figure 4.34 in this range.

In summing up it should be noted that we can look at a large number of rolling maneuvers for different rules of deflection in time of the elevator and the ailerons. However such investigations possess the disadvantage that in them the reaction of the pilot to the motion of the aircraft is not taken into account, which "while interfering" in the control may both improve the characteristics of the transient conditions and thus lower the load acting on the aircraft or make the transient condition worse. Analysis of the dynamics of an aircraft during rolling maneuvers, taking into account the actions of the pilot, may be carried out by investigations in flight or approximately by modeling flight in a trainer. Of certain interest are the investigations of the motions of an aircraft using the mathematical model of a pilot. Only in carrying out such complex investigations is it possible to sufficiently define the most dangerous systems of control of an aircraft and the G-forces acting on them.

CHAPTER V

DYNAMIC CHARACTERISTICS OF AN AIRCRAFT WITH SIMULTANEOUS CONTROL BY AILERONS AND RUDDER

Previously in all sections of this paper we have studied the motions of an aircraft during rolling maneuvers when the aircraft is simultaneously controlled by ailerons and elevator. However, in carrying out certain rolling maneuvers the pilot must deflect the rudder while piloting. Such a deflection of the rudder may be both the result of deliberate action by the pilot causing the maneuver to be carried out or as the result of pilot error. In the present chapter we look at the characteristics of spatial motion of an aircraft with the simultaneous control by ailerons and rudder. We analyze the physical picture of motion and derive the conditions of stability of the controlled motion of an aircraft during a rolling maneuver with deflection of the rudder.

22. Basic Properties of Motion and Stability of an Aircraft with the Simultaneous Control by Ailerons and Rudder.

The motion of an aircraft with simultaneous control by ailerons and rudder possesses the following two characteristics which we shall look at briefly. The first characteristic of spatial motion of an aircraft moving at an angular rolling velocity and with a deflected rudder is that simultaneously with development of the angle of side slip in the aircraft the angle of attack also changes. This phenomenon is similar to that of side slip during rolling of an aircraft flying at a non-zero angle of attack $\alpha_0 \neq 0$ (or $\delta_e \neq 0$), and has the same causes. The physical causes for change in the angle of attack with deflection of the rudder may be simply explained in the following manner. Let us look at the motion of an aircraft rotating relative to the longitudinal axis (OX_1) on which the disturbing moment $\Delta \bar{m}_y = \bar{m}_y^{\delta r} \cdot \delta_r$ from the deflected elevator (Fig. 5.1). The action of the controlling moment $\Delta \bar{m}_y$ on the aircraft leads to the appearance of an angular yawing velocity $\bar{\omega}_y$ and consequently to a deviation of the vector of the total angular velocity $\bar{\Omega}$ from the direction of the major inertial axis of the aircraft OX_1. Because of the nonagreement of the vector of angular velocity with the major inertial axis, centrifugal forces begin to act on the aircraft, which on Figure 5.1 are conditionally shown as being applied to two loads, equivalent to the mass of the

aircraft distributed in the fuselage. It is easy to approximately compute the moment from these forces; it will be equal to

$$M_z^{in} \approx -(J_y - J_x)\omega_y \omega_x. \qquad (5.1)$$
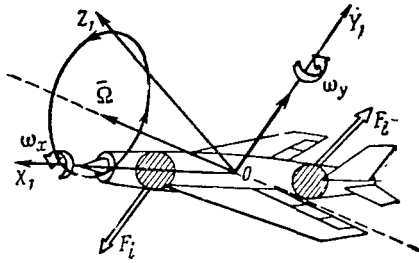
Fig. 5.1

This inertial moment is compensated by the aerodynamic moment of stability $M_z^{\alpha} \Delta\alpha$. From the condition of the equality of the aerodynamic and inertial moment we find an expression for increasing the angle of attack $\Delta\alpha$, which is valid for small angles of the angular rolling velocity ω_x:

$$\Delta\alpha = \frac{-(J_y - J_x)}{m \cdot b_A} \cdot \frac{\omega_x}{V} \frac{c_z^{\beta} m_y^{\delta r}}{m_z^{\alpha} \cdot m_y^{\beta}} \delta_r. \qquad (5.2)$$

The dependence of the steady value in the increase of the angle of attack on the value of the yawing moment $\Delta\overline{m}_y$ is determined by the coefficient $A_m^{\alpha}\delta_r$, the formula for which is shown on Table 2. Using Table 2 we can write an expression for $A_m^{\alpha}\delta_r$ and determine several properties of this function:

$$A_{m\delta_r}^{\alpha} = \frac{\mu^2 \overline{\omega}_x}{A_0}\left(\overline{m}_z^{\overline{\omega}} + A\frac{c_z^{\beta}}{2}\right). \qquad (5.3)$$

From analysis of relationship (5.3) we can make the following conclusions. First, the dependence $A_m^{\alpha}\delta_r(\overline{\omega}_x)$ is an antisymmetric function of the angular rolling velocity ω_x. Secondly, since there are no values in the numerator of Expression (5.3) which are proportional to the critical rolling velocities the form of the function $A_m^{\alpha}\delta_r$ is independent of their relationship, analogous to the manner in which the form of the function $A_m^{\beta}\delta_r$ is independent of the relationship between the critical velocities $\overline{\omega}_{\alpha}$ and $\overline{\omega}_{\beta}$ (see Section 18).

From analysis of the coefficient $A_{m\delta_r}^{\alpha}$ (5.3) it also follows that deflection of the rudder "counter to the rotation", at angular /173 rolling velocities less than the first critical when an angle of side slip is created which inhibits development of the angular rolling velocity, i.e., the following equation is satisfied:

$$\text{sign}\,\delta_r = -\text{sign}\,\overline{\omega}_r, \qquad (5.4)$$

always leads to a negative increase in the angle of attack ($\Delta\alpha < 0$). Deflection of the rudder "in the direction of the rotation", when

the side slip facilitates roll, i.e., the following relationship is valid:

$$\text{sign}\,\delta_r = -\,\text{sign}\,\overline{\omega}_x, \qquad (5.5)$$

always leads to a positive increase in the angle of attack $\Delta\alpha$, regardless of the direction of the roll (the sign of the angular rolling velocity $\overline{\omega}_x$). In fact, satisfaction of condition (5.4) means that

$$\text{sign}\,\overline{\omega}_y = \text{sign}\,\overline{\omega}_x, \qquad (5.6)$$

whence it follows that for all values of the rolling velocity $\overline{\omega}_x M_z^{\text{min}} < 0$ and consequently leads to negative increases in the angle of attack $\Delta\alpha < 0$ (see Fig. 5.1). Analogously we analyze the rolling motion of an aircraft with deflection of the rudder "in the direction of the rotation".

The second characteristic of motion of an aircraft with simultaneous control by ailerons and rudder is the yawing reaction of the aircraft which is opposite in sign to deflection of the rudder δ_r at angular rolling velocities greater than the critical rolling velocities corresponding to the yawing motion. Let us look at this case. From Table 2 we can write the expression for $A_{m\delta_r}^\beta$:

$$A_{m\delta_b}^\beta = \left(\frac{\dot\beta_{ss}}{\Delta\overline{m}_y}\right) = \frac{\mu^2}{A_0}\left[-\left(\overline{m}_{z_1}^\alpha b + \frac{c_y^\alpha\,\overline{m}_z^{\overline{\omega}_z}}{2\mu}b\right) - A\mu\overline{\omega}_x^2\right] \qquad (5.7)$$

In the expression for A_0, if we ignore the terms of damping and cancel, we find an approximate expression for the function $(\beta_{ss}/\Delta\overline{m}_y)$ in the form

$$\left(\frac{\dot\beta_{ss}}{\Delta\overline{m}_y}\right) \simeq \frac{1}{B\mu} \cdot \frac{1}{\left(\overline{\omega}_\beta^2 - \overline{\omega}_x^2\right)} \cdot \qquad (5.8)$$

From Expression (5.8) it follows that at angular rolling velocities which satisfy the inequality $|\overline{\omega}_x| > |\overline{\omega}_\beta|$, the function $(\beta_{ss}/\Delta\overline{m}_y)$ changes sign. Let us recall that the condition

$$|\overline{\omega}_x| < |\overline{\omega}_\beta| \qquad (5.9)$$

is an approximate condition of the aperiodic stability of the yawing motion of an aircraft during a rolling maneuver.

The physical meaning of the change in the relationship between /174 β_{ss} and $\Delta\overline{m}_y$ may be qualitatively explained by the fact that, at angular rolling velocities greater than the critical, the static

stability of yawing motion of the aircraft is disturbed, however the dynamic stability of the total motion is retained. In this case there is a certain analogy with the longitudinal trim of an aircraft when for trim of the stable aircraft we require $\delta_e > 0$, rather than the unstable $\delta_e > 0$.

To explain the results obtained we may approach from another position. The effect on the aircraft of the moment $\Delta \overline{m}_y$ during a rolling maneuver may be studied as a periodic disturbance with a circular frequency ω_x (see Section 18). In this case the increase in ω_x up to values which exceed the critical value ω_β represents passage through resonance and leads to a change by 180° in the reaction phase of the aircraft for β for such a periodic disturbance.

Turning to the precise expression for $(\beta_{ss}/\Delta \overline{m}_y)$, we find that due to the presence of the supplementary terms which depend on damping, in the expression for A_0 when $\overline{\omega}_\alpha > \overline{\omega}_\beta$, the numerator of Expression (5.7) vanishes at values of the angular rolling velocity $\overline{\omega}_x$ smaller than A_0, but when $\overline{\omega}_\alpha < \overline{\omega}_\beta$, this happens at values of ω_x greater than A_0. This leads to a change in the type of the dependence of $A_m^\beta \delta_r$ for the different combinations of $\overline{\omega}_\alpha$ and $\overline{\omega}_\beta$ and causes the respective changes in the dynamics of the aircraft which will be studied below in greater detail. For illustration of these facts on Figure 5.2 we have plotted graphs of the functions $A_m^\beta \delta_r$ for different relationships between the critical rolling velocities $\overline{\omega}_\alpha$ and $\overline{\omega}_\beta$. The dotted lines indicate the curves which correspond to the dynamics of the aircraft not caused by aerodynamic damping.

One of the basic questions in the dynamics of an aircraft in carrying out spatial maneuvers is the question as to maintaining controllability of the aircraft by the ailerons at an angular rolling velocity exceeding the second critical. Losses in controllability of the aircraft are possible in those cases when a stable static solution (singular point) exists which is nonzero when $\Delta \overline{m}_x = 0$, i.e., with undeflected ailerons. Maneuvers for which the aircraft may enter into the condition of motion with angular rolling velocities exceeding the second critical are studied in greater detail in Section 23. In this section we determine the conditions of existence of the solution $\overline{\omega}_x > \omega_{2\,crit}$ when $\Delta \overline{m}_x = 0$ in the general case of carrying out a spatial maneuver by deflection of the ailerons and the rudder at various initial longitudinal trims of the aircraft $(\alpha_0, \Delta\alpha_\delta)$. We conduct the investigation for nonlinear dependence of the lateral moment on the angles of attack and side slip $m_x(\beta, \alpha)$ which can be written in the form

$$m_x = m_x^{\alpha\beta}(\alpha - \alpha^*)\beta,$$

i.e., the lateral moment may be written in the form of a derivative /175 m_x^β which depends on the angle of attack.

Let us look at different extreme variations, when:

(a) $\alpha^* \to 0$, $m_x^{\alpha\beta} < 0$;

(b) $\alpha^* \to 0$, $m_x^{\alpha\beta} > 0$;

(c) $m_x^{\alpha\beta} \to 0$, $m_x^{\alpha\beta}\alpha^* \to m_{x\,0}^{\beta}$ = const.
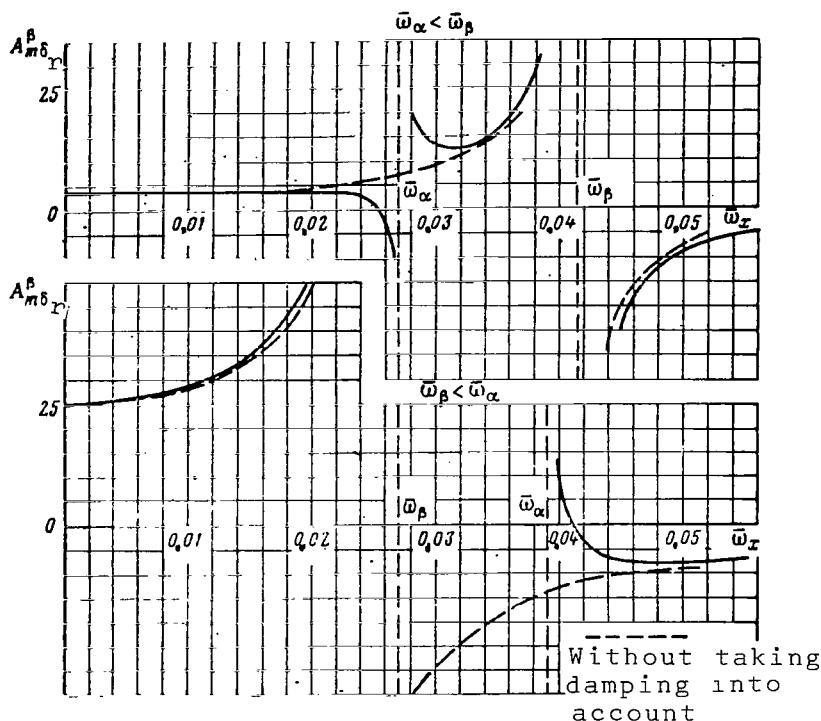
Fig. 5.2

In this case let us not look at the general case when α^* and $m_x^{\alpha\beta}$ are found in an arbitrary relationship. Investigation of this case may be carried out by using the same method which is proposed below but will lead to considerably more awkward computations.

The condition for the existence of a singular point of the equation of motion of an aircraft when $\bar{\omega}_x > \omega_{2\,crit}$ and $\Delta\bar{m}_x = 0$ in the case when $m_x^{\alpha\beta} < 0$, consists of satisfying in the vicinity of this point an inequality, which appears as the condition that the lateral stability of an aircraft must facilitate development of rolling.   /176

$$(\alpha_{s\,s} + \alpha_0 - \alpha^*)\beta_{s\,s} \leqslant 0. \qquad (5.10)$$

The inequality must be satisfied for angular rolling velocities

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$$\bar{\omega}_x \to \bar{\omega}_{2\text{crit}} \left(\bar{\omega}_x \geqslant \bar{\omega}_{2\text{crit}} \right) \qquad (5.11)$$

In the region of angular rolling velocities which satisfy inequality (5.11), assuming that α^* is not too high in Expression (5.10), we can drop the term $(\alpha_0 - \alpha^*)$, which is usually small in comparison to α_{ss} and find condition (5.10) in an approximate form:

$$\alpha_{ss} \beta_{ss} < 0. \qquad (5.12)$$

Inequality (5.12) is satisfied when α_{ss} and β_{ss} have different signs. Using Table 2 let us write expressions for α_{ss} and β_{ss} for a rolling maneuver of an aircraft trimmed at an angle of attack α_0 with deflection of the rudder.

$$\alpha_{ss} = \frac{\mu^2 \bar{\omega}_x}{A_0} \left\{ -\mu A \bar{\omega}_x \left(\overline{m}_y^\beta + B\mu \bar{\omega}_x^2 + \frac{\overline{m}_y^{\bar{\omega}_y} \overline{m}_{z\text{b}}^{\bar{\omega}_z}}{A\mu} \right) \alpha_0 + \right.$$
$$\left. + \left(\overline{m}_{z\text{b}}^{\bar{\omega}_z} + A \frac{c_z^\beta}{2} \right) \Delta \overline{m}_y \right\}; \qquad (5.13)$$

$$\beta_{ss} = \frac{\mu^2}{A_0} \left\{ \bar{\omega}_x \left[\overline{m}_y^{\bar{\omega}_y} \left(\overline{m}_{z\text{b}}^\alpha + \frac{c_y^\alpha \overline{m}_{z\text{b}}^{\bar{\omega}_z}}{2\mu} \right) + AB\mu \bar{\omega}_x^2 \frac{c_y^\alpha}{2} \right] \alpha_0 - \right.$$
$$\left. - \left[\left(\overline{m}_{z\text{b}}^\alpha + \frac{c_y^\alpha \overline{m}_{z\text{b}}^{\bar{\omega}_z}}{2\mu} \right) + A\mu \bar{\omega}_x^2 \right] \Delta \overline{m}_y \right\}. \qquad (5.14)$$

Relationships (5.13) and (5.14) must be studied with angular rolling velocities $\bar{\omega}_x$ that satisfy Condition (5.11) when $A_0 > 0$. It is easy to see that the boundaries of the region which differ in the sign of inequality (5.12) in the coordinates α_0, $\Delta \overline{m}_y$ are straight lines.

$$\left. \begin{array}{l} \alpha_{ss} = 0, \\ \beta_{ss} = 0. \end{array} \right\} \qquad (5.15)$$

The type of the regions (plotted in the coordinates of the parameters of control in the longitudinal and lateral motions (α_0, $\Delta \overline{m}_y$) which possess the property that stable singular points exist in them corresponding to the rolling motion of an aircraft when the ailerons are placed into the neutral position) depends on the relationship between the values of the critical angular rolling velocities of the aircraft $\bar{\omega}_\alpha$ and $\bar{\omega}_\beta$.

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Let us look at the case which is characteristic of flight at subsonic speeds, when $\omega_\beta \ll \omega_\alpha$. For determinancy let us assume that $\omega_x > 0$. Then from Expressions (5.13) and (5.14) it follows that the coefficients for α_0 in both expressions are positive and for $\Delta \overline{m}_y$ they are negative.

Consequently both boundary lines pass through I and II of the quadrant. An example of such regions is given on Figure 5.3. The region of parameters, for which α_{ss} and β_{ss} have different signs and in connection with which loss in controllability of the aircraft by the ailerons is possible, is shown on Figure 5.3 by the shaded areas. From Figure 5.3. it is clear that this region does not encompass the straight line $\Delta \overline{m}_y = 0$ which corresponds to motion of an aircraft with the rudder in neutral position (see Section 19). The region itself in which loss of controllability of the aircraft by the ailerons is possible is quite narrow and a rather precise agreement is necessary between α_0 and $\Delta \overline{m}_y$ in order that when the ailerons are placed in the neutral position the aircraft does not cease rotation. Hence in particular we can make the rather important conclusion that when $\omega_\beta \ll \omega_\alpha$ the aircraft practically always retains controllability of the ailerons even in that case when the rudder during the entire time of the maneuver is deflected and at the end of the maneuver is not placed into the neutral position.
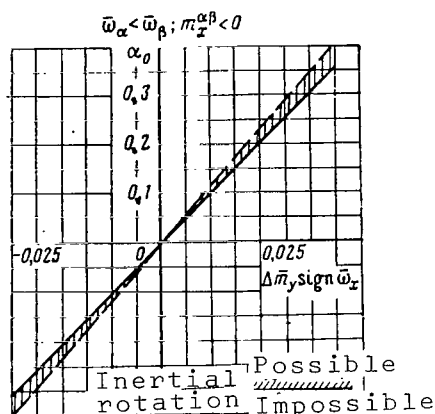
Fig. 5.3

Let us proceed to an analysis of the stability of motion of an aircraft in the vicinity of the second critical rolling velocity with a relationship of the critical velocities $\omega_\alpha \gg \omega_\beta$, characteristic for flight at supersonic speeds. In this case the straight lines $\alpha_{ss} = 0$ and $\beta_{ss} = 0$ are arranged in the II and IV quadrants (Fig. 5.4). From Figure 5.4 it is obvious that for the relationship of the critical rolling velocities when $\omega_\alpha \gg \omega_\beta$, the picture is substantially different than with the reverse sign of the inequality. The region in which losses are possible in the controllability of the aircraft now is encompassed completely in quadrants I and III and the greater part of quadrants II and IV, including the line $\Delta \overline{m}_y = 0$. Hence it follows that practically in all combinations of α_0 and $\Delta \overline{m}_y$ during rolling maneuvers accompanied by the development of angular rolling velocities exceeding in value the second critical, the aircraft may lose controllability by the ailerons (with values of α_0 and $\Delta \overline{m}_y$ from the shaded region on Fig. 5.4).

For a more complete representation let us look briefly at the cases when the lateral stability of an aircraft is independent of

the angle of attack, i.e., $m_x^{\alpha\beta} = 0(m_{x0}^{\beta} < 0)$. The condition of
loss of stability, just as before, is such an influence of the
lateral stability on the motion of the aircraft which facilitates
development of angular rolling velocity of the aircraft. When
$\bar{\omega}_x > 0$, for this the following inequality is satisfied:

$$\beta_{ss} \leqslant 0: \qquad (5.16)$$

$$\bar{\omega}_x > \bar{\omega}_b, m_x^{\alpha\beta} < 0$$
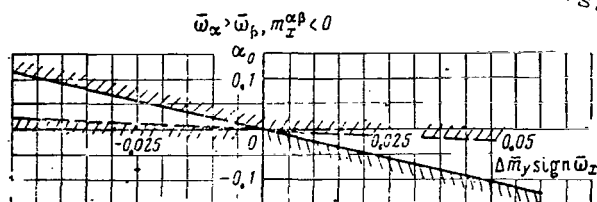
Fig. 5.4

Using Relationship (5.14)
it is easy to construct the
boundaries of the regions both
for the case $\omega_\beta \ll \omega_\alpha$ and for
$\omega_\beta \gg \omega_\alpha$, which defines the values
of the parameters of control
$(\alpha_0, \Delta\bar{m}_y)$ for which realization
of systems of inertial rotation

of an aircraft is possible. Examples of the regions are graphically
shown on Figures 5.5 and 5.6. The characteristic singularities of
the motion of an aircraft caused by lateral stability which is
independent of the angle of attack is that, when $\Delta\bar{m}_y = 0$, the entry
of the aircraft into a system of inertial rotation regardless of
the relationship between the critical angular velocities $\bar{\omega}_\alpha$ and $\bar{\omega}_\beta$
is possible only during rolling maneuvers of the aircraft trimmed
at a negative angle of attack.

Finally if $m_x^{\alpha\beta} > 0$ and α^* is not very high, the condition of
loss of controllability of the aircraft by the ailerons at high
rolling velocities can be written analogous to expression (5.12),
only the sign of the inequality must be changed to the positive

$$\alpha_{ss}\beta_{ss} > 0 \qquad (5.17)$$

It is obvious that in this case the boundaries of the regions
agree with those plotted on Figures 5.3 and 5.4, only now the
region is shaded where earlier it was not (Figs. 5.7 and 5.8).
From these figures it follows that in the case when $m_x^{\alpha\beta} > 0$ i.e.,
it has a sign which is opposite to the usual for subsonic flying
speeds, from the point of view of the controllability by the ailer-
ons at angular rolling velocities exceeding the value of the second
critical $\bar{\omega}_x > \omega_{2crit}$, the relationship $\bar{\omega}_\beta \gg \bar{\omega}_\alpha$ is not favorable.

23. The Effect of Rudder Deflection on the Dynamics of an Aircraft During Rolling Maneuvers

In this section we look briefly at the characteristics of the
transient conditions of an aircraft during rolling maneuvers carried
out with the simultaneous gradual deflections of ailerons and rud-
der for two basic original conditions of flight (horizontal flight
and flight with a negative G-force). Certain characteristics of
the motion of an aircraft with control by ailerons and rudder were
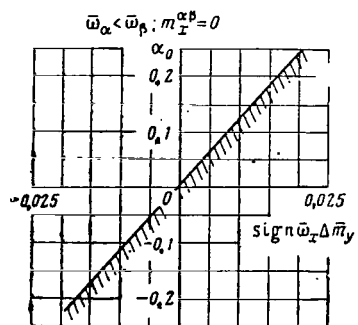
Fig. 5.5
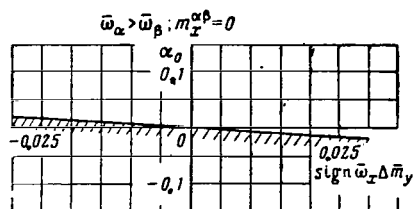
Fig. 5.6
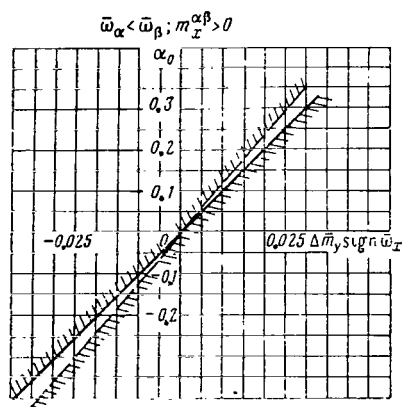
Fig. 5.7
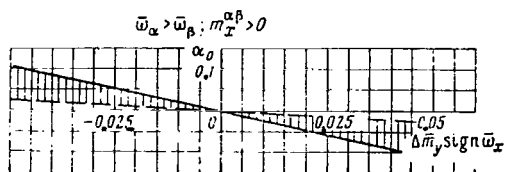
Fig. 5.8

studied in Section 22. In the present section the basic attention
is paid to an analysis of the possibility of the escape of an air-
craft into large angular rolling velocities, in particular into
angular velocities which exceed the second critical, and to an
investigation of the controllability of an aircraft by ailerons d
during rotation at such angular velocities. In the investigations
we will take into account the dependence of the derivatives of the
moment of lateral stability m_x^β on the angle of attack, i.e., we
shall assume that the function $m_x^\beta(\alpha)$ has a negative derivative for
the angle of attack $(\partial m_x^\beta/\partial\alpha < 0)$. Such a dependence of the lateral
stability on the angle of attack includes a number of character-
istics for the dynamics of the aircraft which were partially men-
tioned in Sections 19 and 22. The most substantial influence of the
dependence of the lateral stability on the angle of attack appears
in those cases when in the process of the rolling maneuver the
angles of attack of the aircraft become negative. In fact in this
case the sign of the derivative $m_x^\beta(\alpha)$ changes and the influence of
the angle of side slip changes to the opposite. A characteristic
example of such a motion is the rolling maneuver of an aircraft
with simultaneous control by the rudder "counter to the rotation",
i.e., when

$$\text{sign } \delta_r = -\text{sign } \overline{\omega}_x.$$

Deflection of the rudder δ_r during a rolling maneuver counter
to the rotation leads to the development of an angle of side slip
of the aircraft, which prevents the rolling, and to a decrease in
the angle of attack and with a sufficiently large value of the rud-
der deflection to the escape of the aircraft in the process of the
rolling maneuver to negative angles of attack. With negative angles
of attack there occurs a change in the sign of the lateral stability
$m_x^\beta(\alpha)$ (see Fig. 4.9) and if previously the side slip inhibited
development of rolling then the reverse phenomenon occurs now, i.e.,
the side slip of the aircraft begins to facilitate development of
roll. The aircraft seems to "catch-up" and the angle of rolling
velocity begins to grow until it no longer exceeds the value of the
second critical velocity.

This section consists of two basic sections in which we analyze
the effect of rudder deflection counter to the rotation and in the
direction of the rotation on the dynamics of the aircraft during
rolling maneuvers. In each section the investigation is carried
out both for the case when the smaller critical rolling velocity is
$\overline{\omega}_\alpha(\overline{\omega}_\alpha \ll \overline{\omega}_\beta)$ and for the case when the smaller is $\overline{\omega}_\beta(\overline{\omega}_\alpha \ll \overline{\omega}_\beta)$.

1. Rolling Maneuvers of an Aircraft with Simultaneous Deflection of the Rudder Counter to the Direction of Rotation (sign δ_r = -sign$\overline{\omega}_x$)

(a)   The case $\overline{\omega}_\alpha \ll \overline{\omega}_\beta$ (the relationship is characteristic for
M < 1 numbers). The character of the change in the derivatives of
the static solutions for the basic parameters of motion of the air-
craft in this case were studied in great detail above and is obvious

from Figures 5.9 and 5.10. For a rolling maneuver carried out from
the conditions of horizontal flight, examples of the static solu-
tions for the basic parameters of motion are graphically shown on
Figures 5.11 - 5.13, and the transient conditions on Figure 5.14.
For these graphs we must make the following explanations. The
deflection δ_r counter to the direction of rotation when $\overline{\omega}_x > 0$ is
carried out to the negative side ($\delta_r < 0$ on all the graphs for the
static solutions) and when $\overline{\omega}_x < 0$ to the positive direction ($\delta_r > 0$
on all graphs with transient conditions). This agreement must be
taken into account in comparing the static solutions and the tran-
sient conditions.
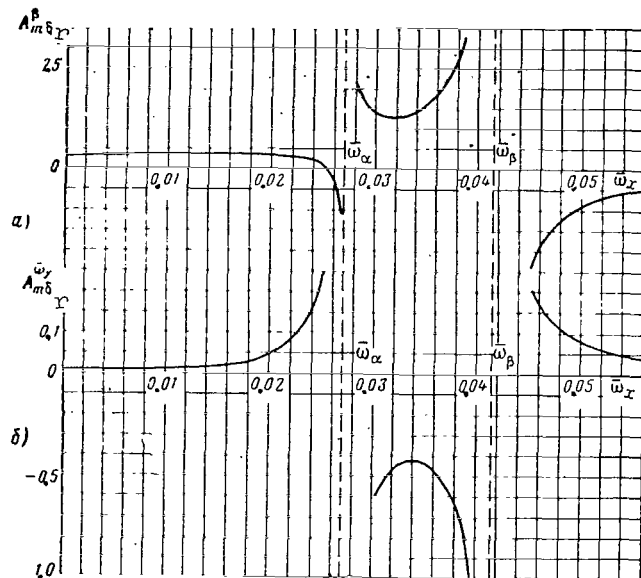
Fig. 5.9
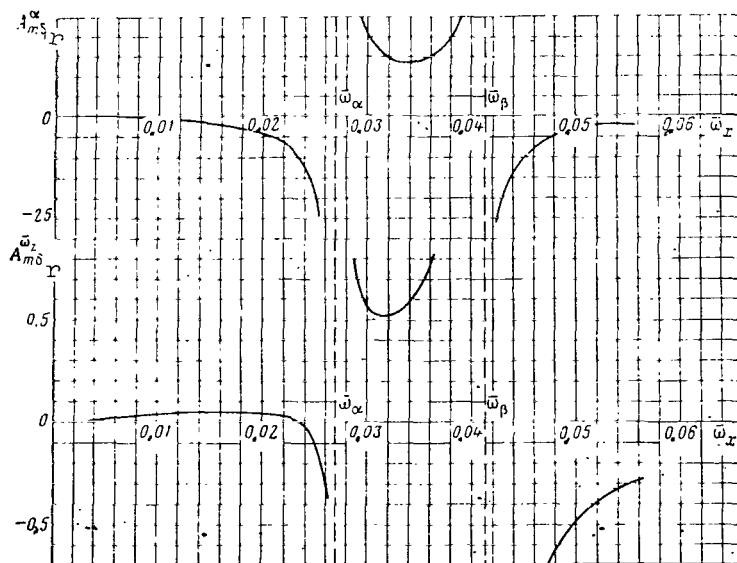
Fig. 5.10
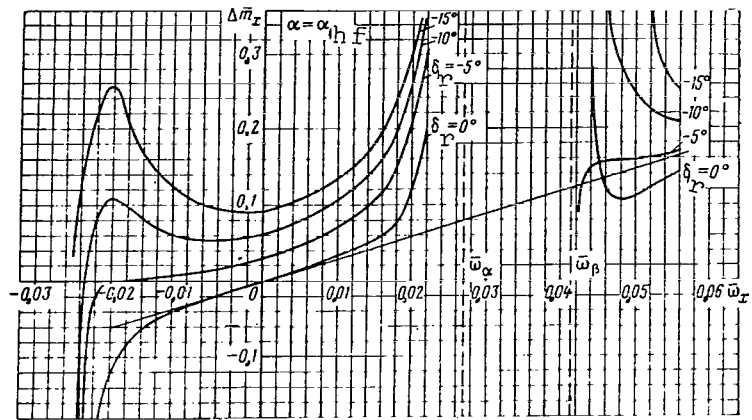
Fig. 5.11
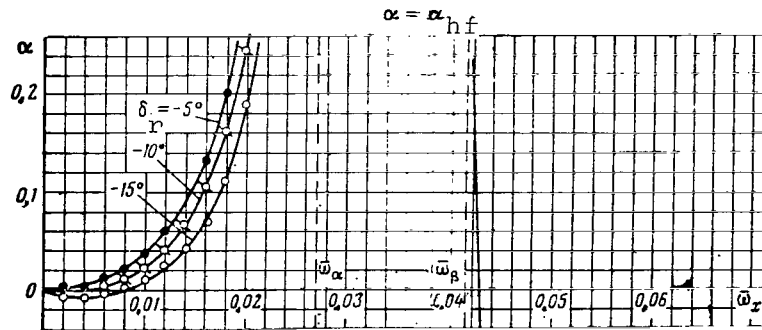
Fig. 5.12
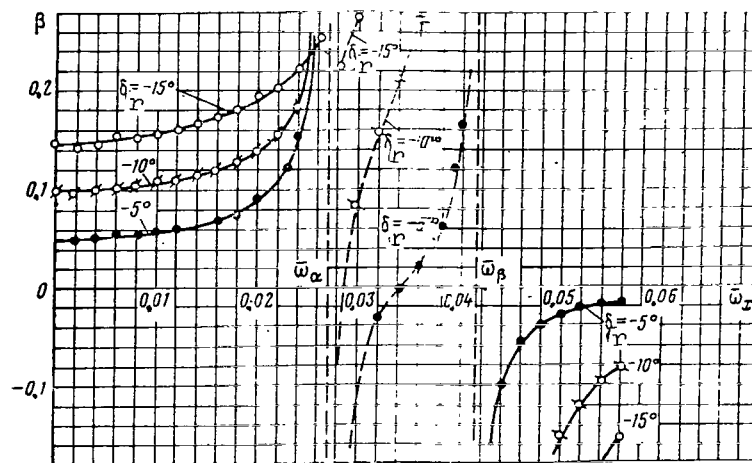
Fig. 5.13

The characteristics of the rolling maneuvers, with the simultaneous deflection of the rudder counter to the direction of rotation from the conditions of flight with a positive normal G-force, are caused by the fact that the influence of the trim angle of attack α_0 and the moment from the rudder Δm_y leads to an increase in the angle of attack α of different signs and the angle of side slip β of identical signs. In this respect with the rolling maneuvers with small aileron deflections due to the large lateral stability of the aircraft $m_x^\beta(\alpha)$ a basic effect is exerted on the motion of the aircraft by the deflection of the rudder δ_r, which, as a result
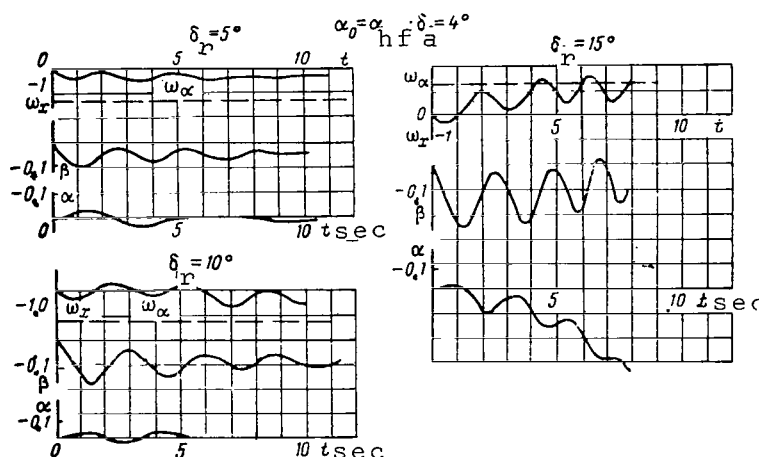
Fig. 5.14

of the development of a large angle of side slip of the aircraft, may even lead to a change in the direction of the roll. On the other hand as noted above during rolling maneuvers with large deflections of the ailerons and rudder counter to the direction of rotation it is possible for the aircraft to go into negative angles of attack. With the motion of the aircraft with a negative angle of attack there occurs a change in the sign of the lateral stability $m_x^\beta(\alpha)$ due to which the side slip that previously inhibited development of an angular rolling velocity of the aircraft now begins to facilitate its growth until ω_x no longer exceeds the value of the second critical rolling velocity (Fig. 5.15). Maneuvers of this type correspond to the escape of the aircraft into a system of inertial rotation.

Graphs of the static solutions for the basic parameters of an aircraft during a rolling maneuver with the simultaneous deflection of the rudder counter to the direction of rotation from the conditions of flight with a negative G-force are shown on Figures 5.16-5.18. For rolling maneuvers carried out under these conditions it is characteristic that there be the presence of stability of the branch of the static solution $\Delta \overline{m}_y(\overline{\omega}_x)$ at an angular rolling velocity less than the first critical velocity which depends quite weakly on

Fig. 5.15

Fig. 5.16

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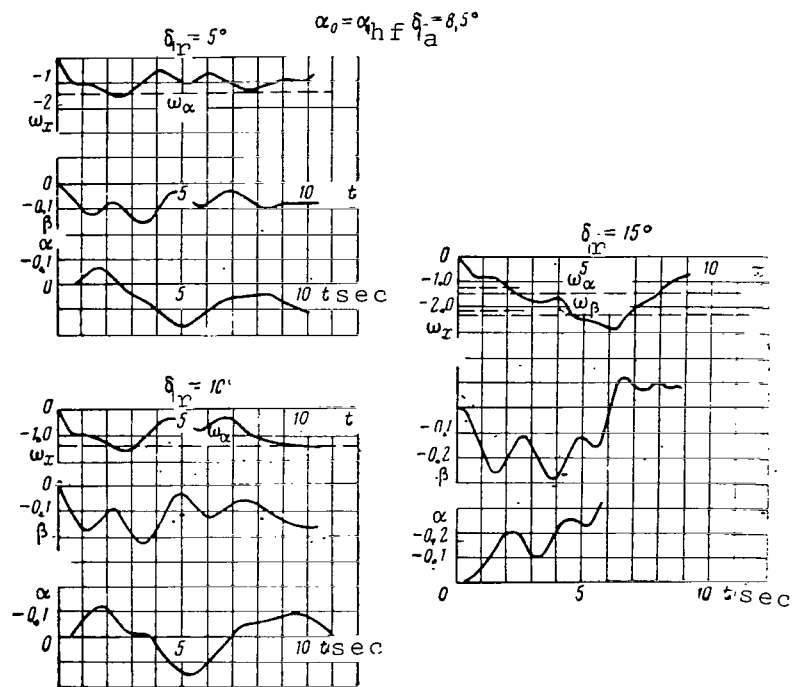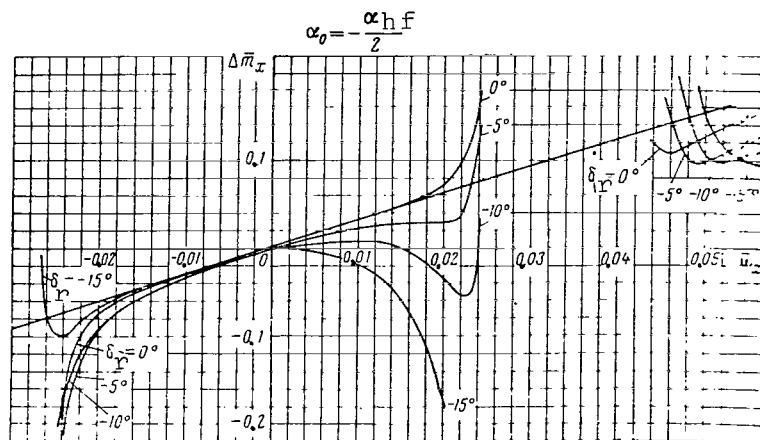
the values of the deflections of the ailerons and the rudder. In
connection with this the angular rolling velocity of the aircraft
with slow deflection of the ailerons and rudder practically is

$$\alpha_0 = -\frac{\alpha_h f}{2}$$

Fig. 5.17

$$\alpha_0 = -\frac{\alpha_h f}{2}$$

Fig. 5.18

$$\alpha_0 = -\frac{\alpha_h f}{2}; \delta_3 = 8,5^\circ$$
$$\delta_{r} = 5^\circ$$
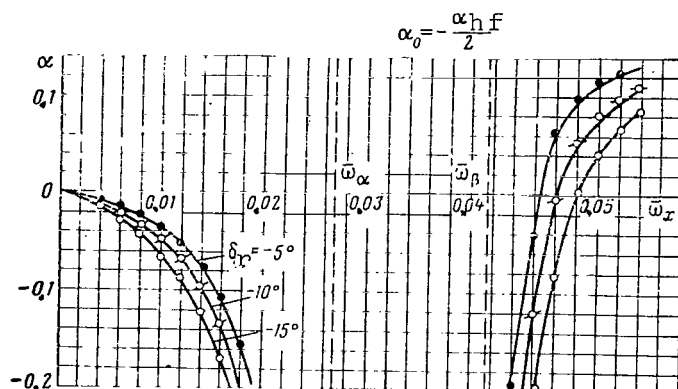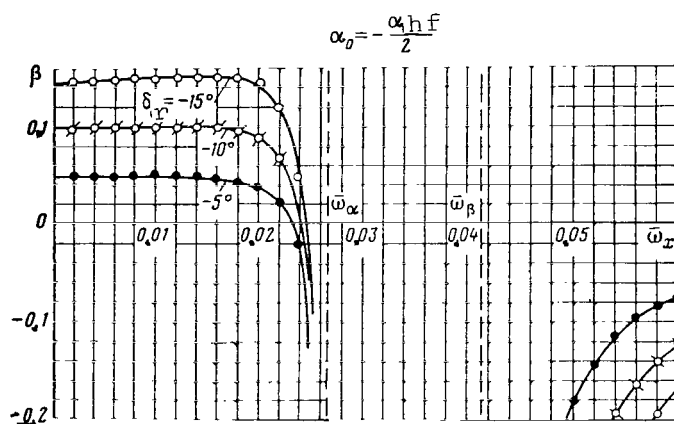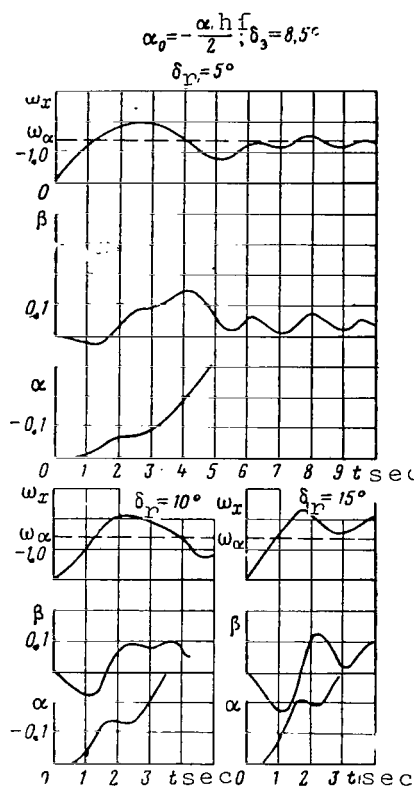
Fig. 5.19

independent of the value of the deflections. Examples of the
transient conditions according to the basic parameters of motion
are shown on Figure 5.19. Let us note that with the approach of
the value of the angular rolling velocity to the value of the first
critical velocity, the angle of side slip of the aircraft changes
sign to the opposite (regardless of the fact that $\Delta \overline{m}_y$ sign $\overline{\omega}_x < 0$,
the angle of side slip β becomes positive) and begins to inhibit
the development of rolling. Because of this, at small elevations
of the ailerons, no angular rolling velocities greater than the
second critical are seen. With the motion of an aircraft at large
angular rolling velocities ($\omega_x > \omega_{x2crit}$), which is reached by

large and sharp deflections of the ailerons, placing the ailerons
into the neutral position stops the rotation of the aircraft even
with the rudder kept in the deflected position.

(b) *The case* $\bar{\omega}_\alpha \gg \bar{\omega}_\beta$ (the relationship is characteristic for
numbers $M > 1$). Examples of the change in the derivatives of the
static solutions for such relationships of the critical velocities
are shown on Figures 5.20 and 5.21. Examples of the static solutions
for the basic parameters of motion of an aircraft during rolling
maneuvers carried out from conditions of horizontal flight are
shown on Figures 5.22 - 5.24. Analysis of the transient conditions
of an aircraft and of the static solutions shows that with small
deflections of the ailerons the rolling motion of the aircraft is
determined by the value of the deflection of the rudder (Fig. 5.25).
The fact is characteristic that the beginning of the rolling
maneuver is accompanied by a strong change in the angle of side
slip so that in a number of cases in the analyzed examples the angle
of side slip β exceeds 30°. Such a change in the angle of side
slip is caused by the small degree of yawing stability of the air-
craft. With the escape of the aircraft into angular rolling veloc-
ities exceeding the value of the second critical velocity and the
ailerons placed into a neutral position, losses may occur in the
controllability of the aircraft by the ailerons which is expressed
in the retention of a practically constant value of the angular
rolling velocity regardless of the fact that the ailerons are lo-
cated in an undeflected position.

Rolling maneuvers with a negative initial normal G-force are
characterized by the relative ease of escape of the aircraft into
an angular rolling velocity exceeding the value of the second
critical. The static solutions and the transient conditions for   /189
this case of the maneuvers are shown on Figures 5.26 - 5.28. Placing
the ailerons into a neutral position usually does not stop the
spinning rotation of the aircraft. Rotation of the aircraft is
stopped, and thanks to the influence of δ_r, the direction of the
roll may even be changed, if at the moment that the ailerons are
placed into the neutral position the conditions $(\alpha_{ss} \cdot \beta_{ss} > 0)$ are
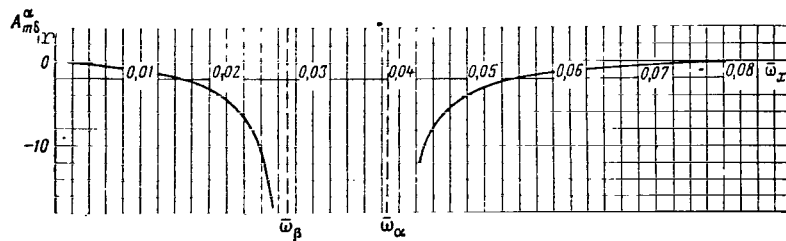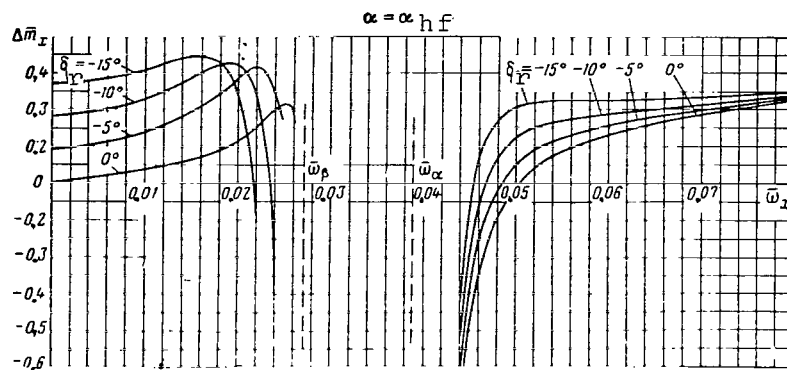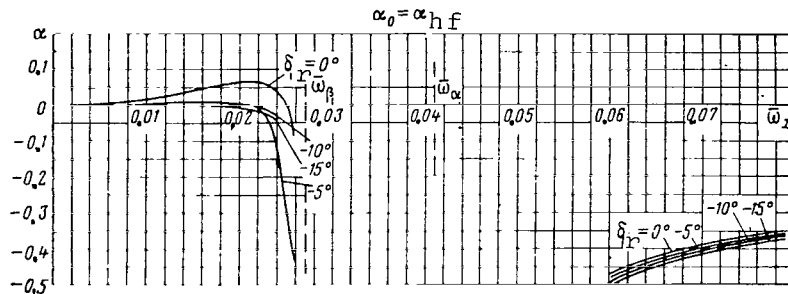satisfied.
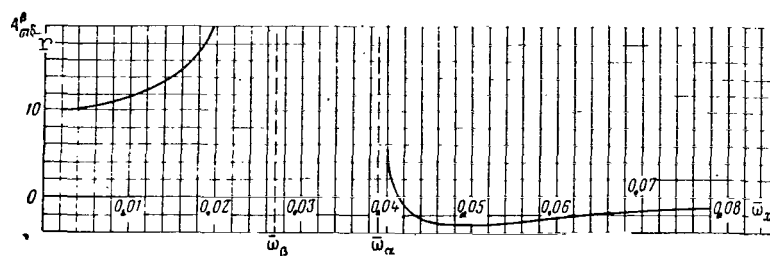
Fig. 5.20

Fig. 5.21
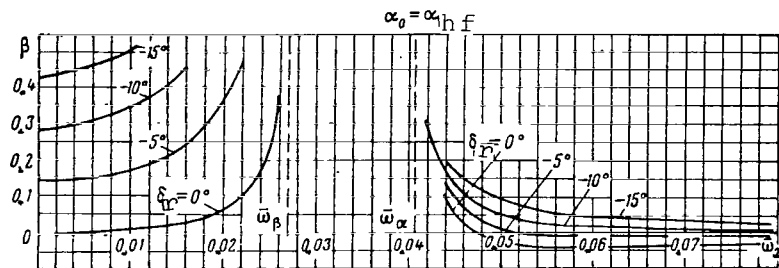

Fig. 5.22

Fig. 5.23

Fig. 5.24

Rolling maneuvers when the ailerons are placed into the neutral position are usually accompanied by a strong development of angles of side slip β and attack α (see Fig. 5.28).

2. Aileron Control with Simultaneous Deflection of the Rudder in the Direction of Rotation (sign δ_r = sign $\overline{\omega}_x$)

(a) The case $\overline{\omega}_\alpha \ll \overline{\omega}_\beta$ (the relationship is characteristic for numbers $M < 1$),

Examples of static solutions for the basic parameters of motion of an aircraft during rolling maneuver with a simultaneous deflection of the rudder in the direction of rotation from conditions of horizontal flight are shown on Figures 5.29 - 5.31. The characteristic feature of the dependence of the size of the required moment of the ailerons on $\overline{\omega}_x[\Delta m_x(\overline{\omega}_x)]$ is that the angular rolling velocity with small aileron deflections to a very large degree is determined by the size of the rudder deflection but, beginning from certain values of the deflection δ_r and the approach of the value of the angular rolling velocity to the value of the first critical velocity, generally ceases to depend on the deflections δ_r and δ_a. Such a small dependence of the angular rolling velocity on the size of the control deflection is explained by the fact that with the approach of the rolling velocity ω_x to the critical value the angles of attack and side slip of the aircraft begin to grow strongly and there is a rather small change in the angular rolling velocity in order for the lateral moment $[m_x^\beta(\alpha)\cdot\beta]$ to compensate the changes in the values δ_a and δ_r.

In the region of angular velocities of rotation of the aircraft for rolling which exceeds the value of the second critical veloci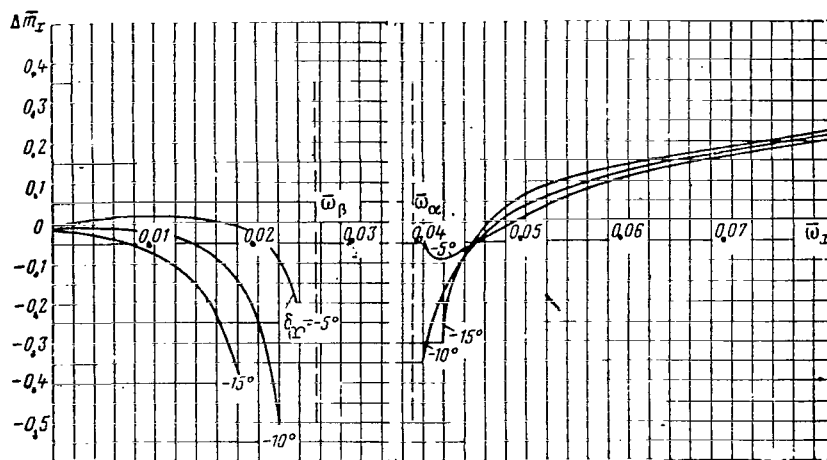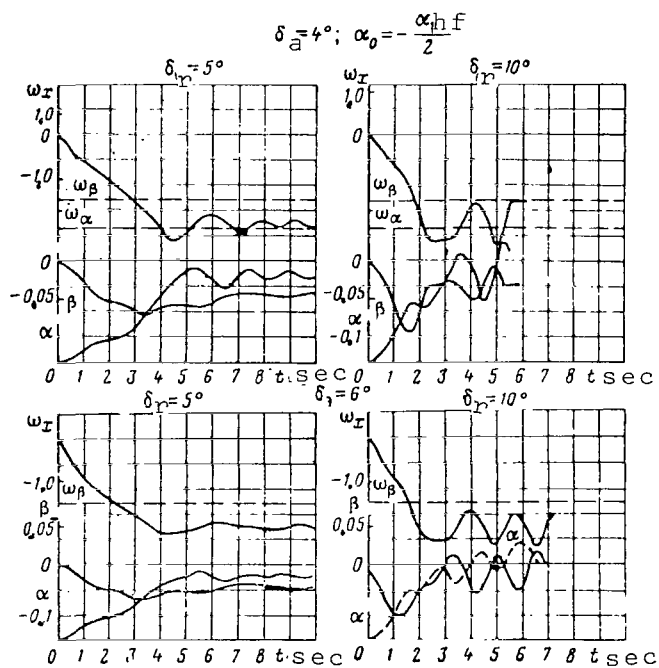ty ($\omega_x > \omega_{x2\,crit}$), the ailerons is retained deflections of the rudder and of the aircraft (α_0) (see Section 22).
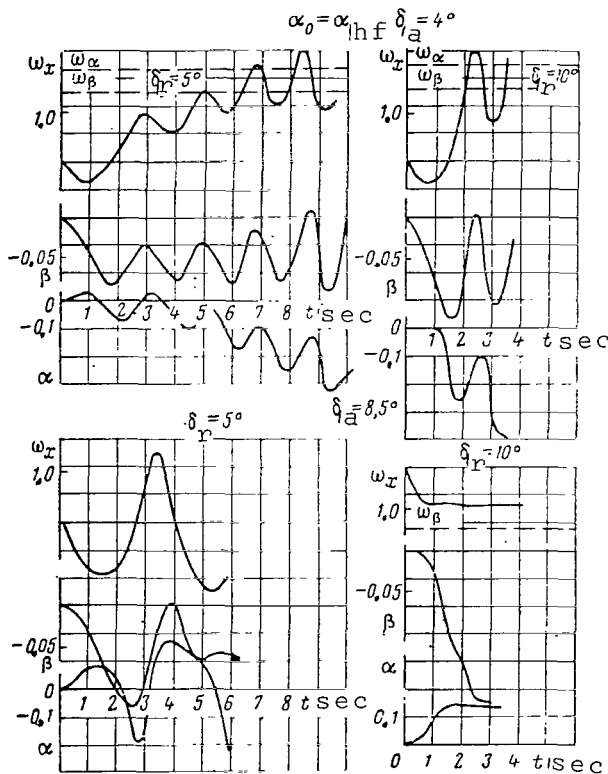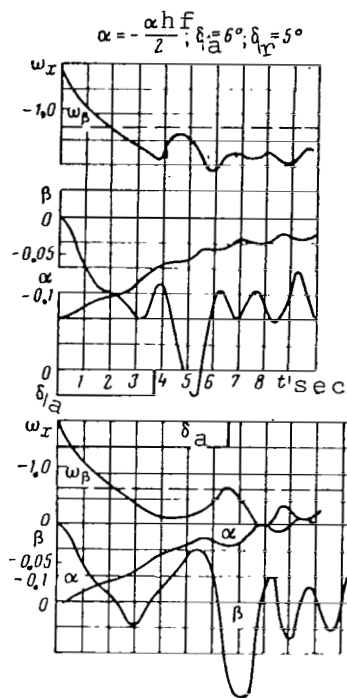

Fig. 5.25

controllability of the aircraft by the ailerons practically in all combinations of conditions of longitudinal trim of the aircraft (see Section 22).

Fig. 5.26

Fig. 5.27

$$\alpha = -\frac{\alpha h f}{2}; \delta_a = 6°; \delta_r = 5°$$

Fig. 5.28

$$\alpha = \alpha_{h f}$$
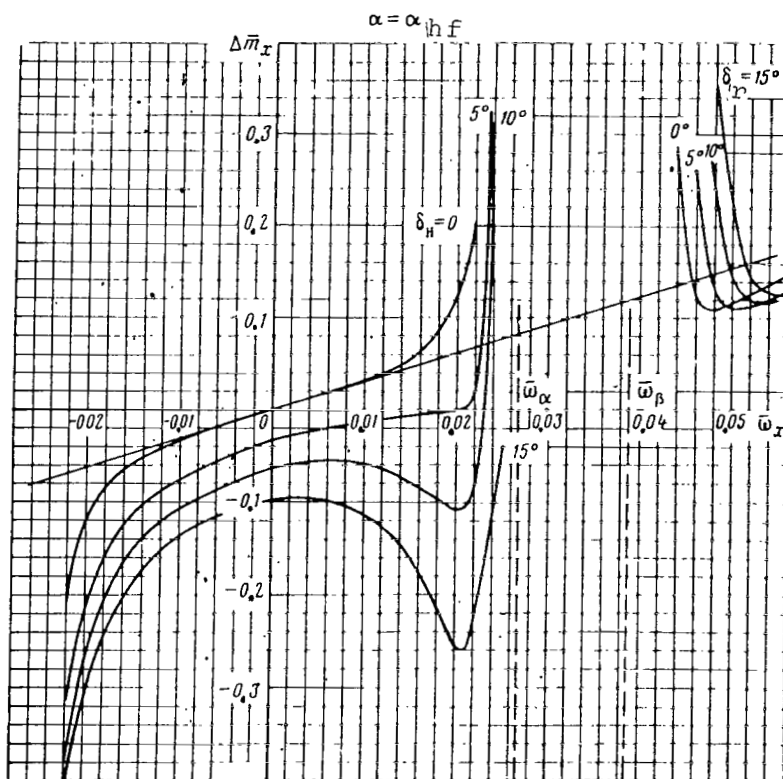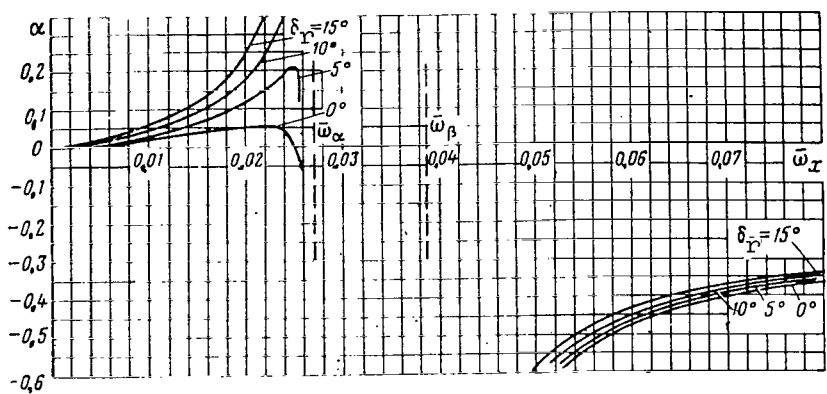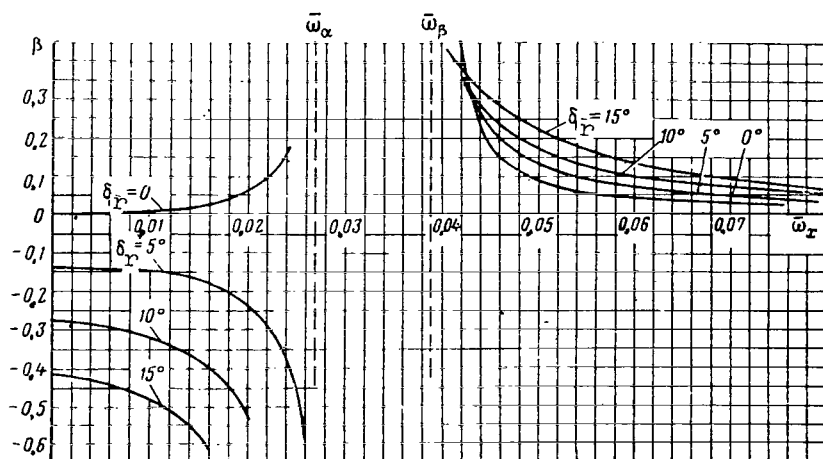

Fig. 5.29

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Fig. 5.30

Fig. 5.31

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Examples of the transient conditions for the basic parameters of   /194
motion of an aircraft with gradual deflection on the controls are
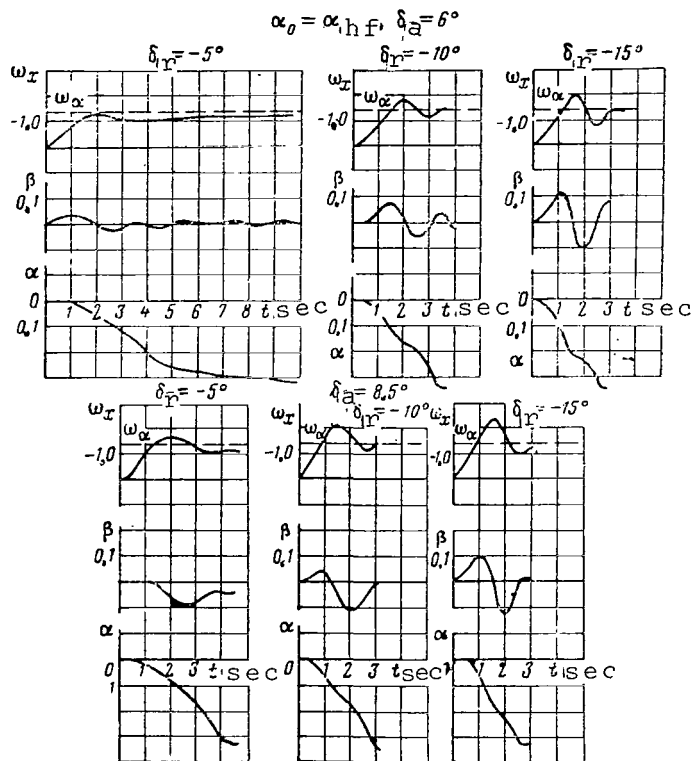shown on Figures 5.32 and on Figure 5.33 is shown the graph of change

Fig. 5.32

in parameters of motion of an aircraft with the ailerons placed in-
to the neutral position and the rudder kept in a constant position.
From Figure 5.33 it is clear that after placing the ailerons into
the neutral position, an angular rolling velocity is retained that
is determined by the value of the rudder deflection.

Examples of the static solution for the basic parameters of
motion of an aircraft during a rolling maneuver carried out from
conditions of flight with a negative normal G-force are shown on   /195
Figure 5.34 and examples of the transient conditions on 5.35. The
characteristic feature of these rolling maneuvers is the relative
simplicity of escape of the aircraft into an angular rolling veloc-
ity exceeding the value of the second critical velocity. An inter-
esting property of the rolling maneuver carried out at a longitudinal
trim of the aircraft with an angle of attack $\alpha_0 = \alpha_{hf}/2$, is that
when $\Delta \overline{m}_x = 0$ regardless of the deflection of the rudder in the
direction of rotation the motion of the aircraft occurs at a small
angular rolling velocity (see Fig. 5.34). This fact is explained

by the small value of the lateral stability of the aircraft $m_x^\beta(\alpha)$ in the examined example with a trim angle of attack $\alpha_0 = \alpha_{hf}/2$, which makes the rudder practically ineffective in creating an angular rolling velocity.
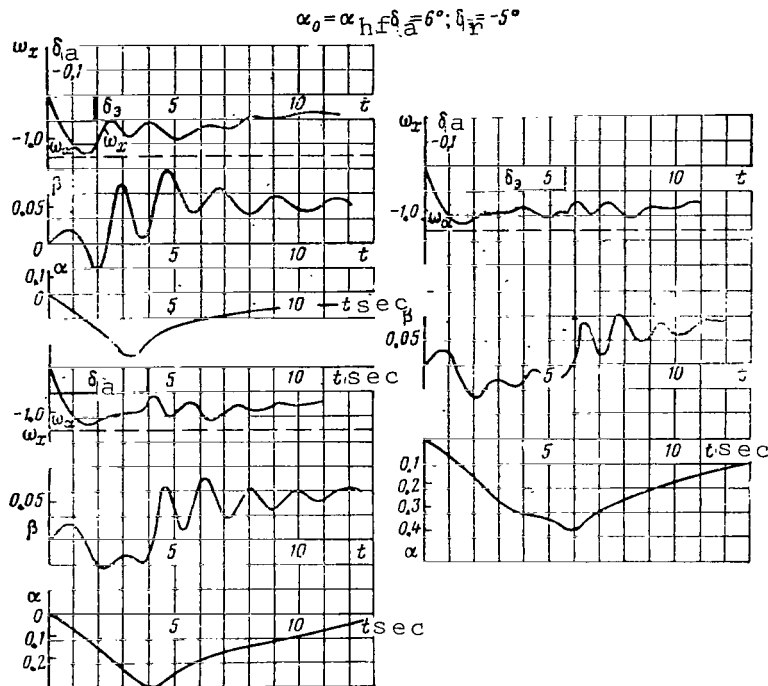
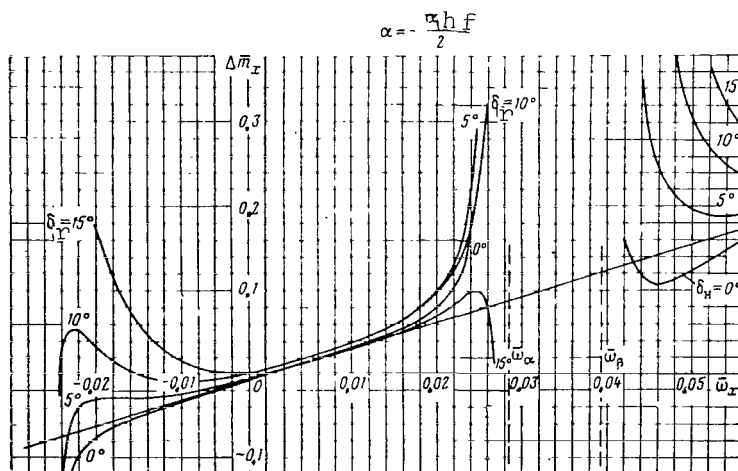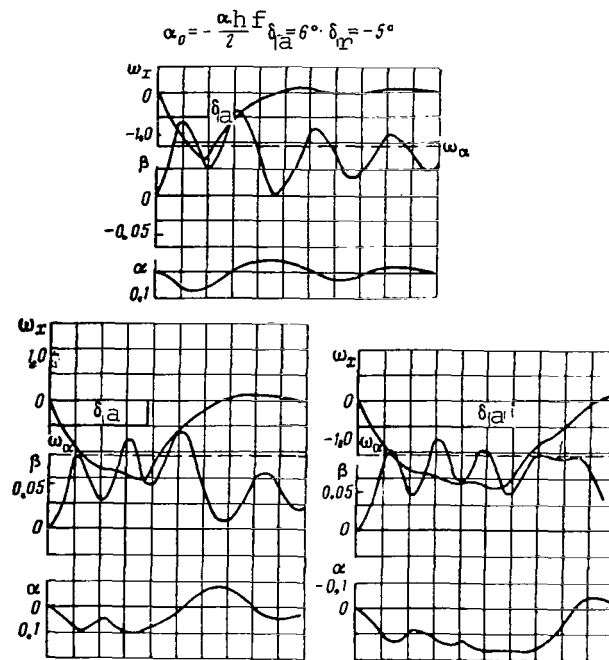$$\alpha_0 = \alpha_{hf} \quad \delta_a = 6°; \quad \delta_r = -5°$$

Fig. 5.33

$$\alpha = -\frac{\alpha_{hf}}{2}$$

Fig. 5.34

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$$\alpha_0 = -\frac{\alpha_1 h f}{2} \quad \delta_{|a} = 6°, \quad \delta_n = -5°$$
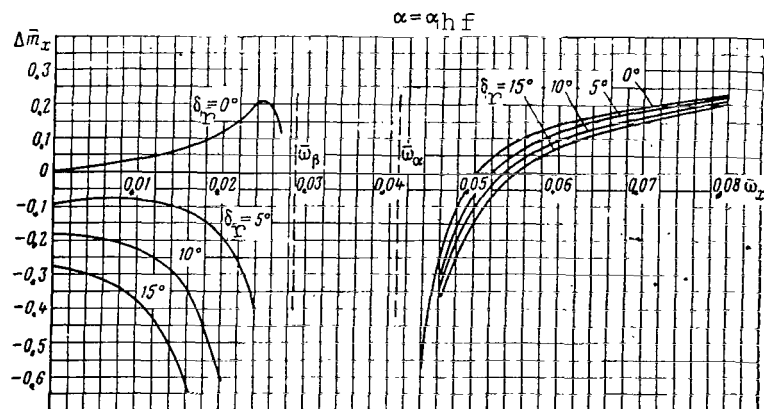

Fig. 5.35

$$\alpha = \alpha_1 h f$$

Fig. 5.36

Recordings of the transient conditions with the ailerons placed into neutral position and retention of the rudder in a deflected position are shown on Figure 5.35.

(b) *The case* $\bar{\omega}_\alpha \gg \bar{\omega}_\beta$ (the relationship is characteristic for numbers $M > 1$). Examples of the static solutions for the basic parameters of motion of an aircraft during a rolling maneuver with the simultaneous deflection of the rudder from conditions of horizontal flight and flight with a negative G-force are shown on Figure 5.36 and 5.37. The feature of the controlled motion of an aircraft in these cases is the relative simplicity of the escape of the aircraft into angular rolling velocities exceeding the second critical velocity. Examples of the transient conditions for maneuvers carried out from conditions of horizontal flight are shown on Figure 5.38. From the figure it is obvious that placing the ailerons into a neutral position will not stop the rotation of the aircraft. Examples of the transient conditions for the maneuvers carried out from conditions of flight with a negative G-force are shown on Figure 5.39, in particular, in placing the ailerons into a neutral position. From Figures 5.38 and 5.39 it follows that placing the ailerons into a neutral position in this case will not stop the rolling turn of the aircraft, i.e., there is observed a motion which is called the inertial rotation of the aircraft. It must be noted that motion with the loss of stability usually exists in carrying out a large number of rolling turns of an aircraft. In carrying out one turn such systems usually are unable to develop and the aircraft retains controllability.
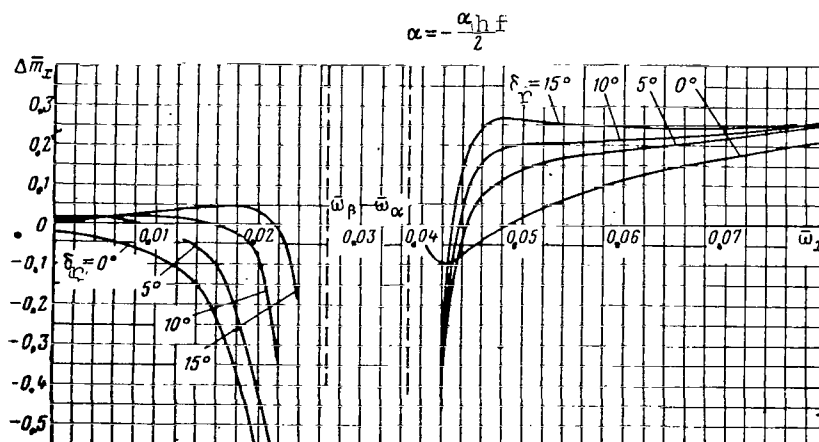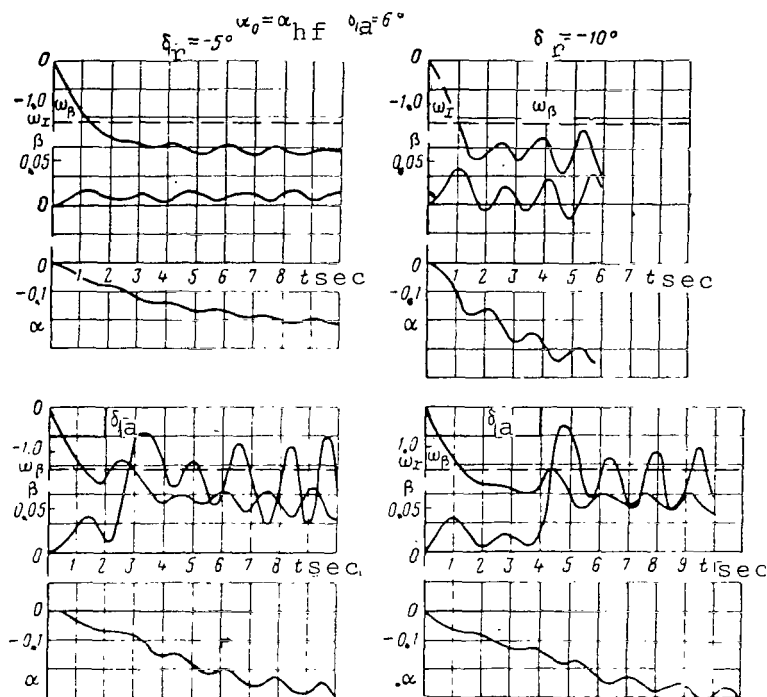
Fig. 5.37

Fig. 5.38

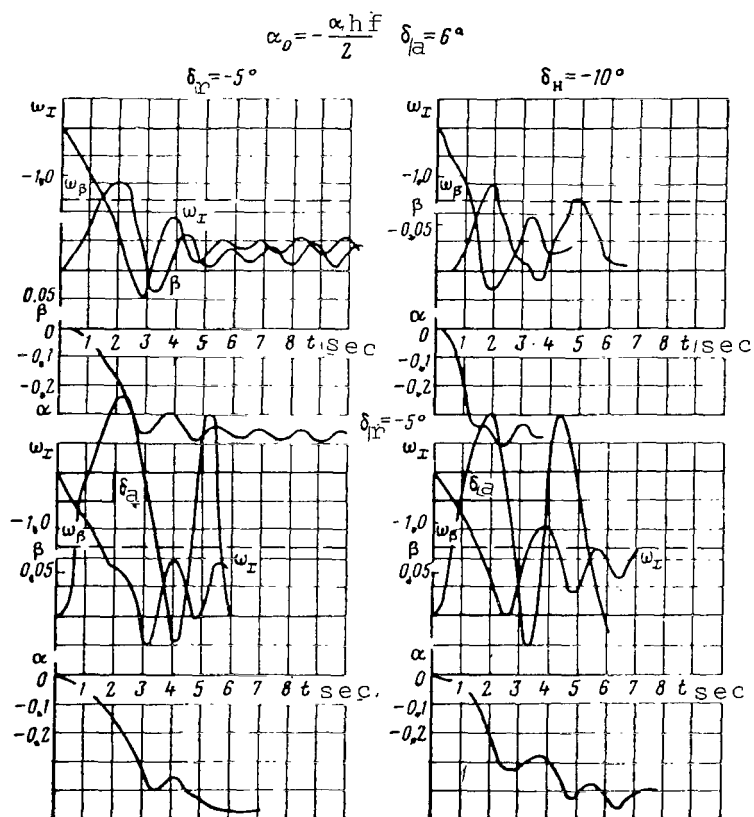Fig. 5.39

CHAPTER VI

THE EFFECT OF AUTOMATIC INSTRUMENTS ON THE DYNAMICS OF AN AIRCRAFT DURING SPATIAL MANEUVERS

The characteristics of the spatial maneuvers, leading in cer- /200
tain instances to the development of motion of an aircraft accom-
panied by significant G-forces and angular velocities, causes us to
seek methods for improving the dynamic characteristics of modern
aircraft. A solution to this problem is possible in three basic
directions:

1. Guaranteeing the proper inertial and aerodynamic character-
istics of the aircraft in the design.

2. Introducing limitations to the piloting of the aircraft.

3. Using automatic equipment.

Each of these directions has positive and negative sides. It
is obvious that we are far from being able to guarantee the required
(from the point of view of spatial motion) aerodynamic, or even the
inertial, characteristics of an aircraft. Limitations to the
piloting in certain instances make the maneuvering capabilities of
the aircraft worse as well as not being unallowable in all cases.
The most promising means is the use of automatic equipment for im-
proving the dynamic characteristics of the aircraft.

Several variations of automatic systems which permit improving
the dynamics of an aircraft during spatial maneuvers have been
studied in the special literature [48], [55], however this problem
can never be assumed as being sufficiently developed and we shall
not remain on it for any period of time. The material introduced
below is devoted only to one specific problem of automation.

We shall analyze the dynamics of the spatial motion of an air-
craft equipped with oscillation dampers relative to all three major
inertial axes.

And finally in the last section of this chapter we look briefly
at several requirements for the aerodynamic characteristics of air-
craft and discuss arguments as to the possible requirements on
limitation of the maneuvers.

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24. The Effect of Oscillation Dampers on the Dynamics of an Aircraft During Rolling Maneuvers.

At the present time in the majority of modern maneuvering air-
craft automatic means are widely used to artificially stabilize and
raise the dynamic stability (oscillation dampers along the pitching
channels, the course channels, and the rolling channels. In con-
nection with this let us look briefly at the influence of such
damping devices on the dynamics of an aircraft during spatial
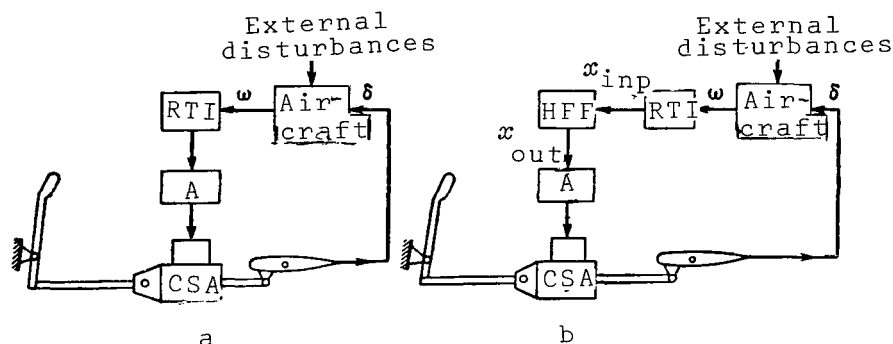maneuvers with strong rolling.

Fig. 6.1

Figure 6.1 shows the principal schematics of two of the most
widely used types of dampers. Figure 6.1,a shows the principal
block schematic of the simplest damper composed of a rate-of-turn
indicator (RTI), a signal amplifier indicator (A), and a control-
surface actuator (CSA). The damper operates in the following way.
With the appearance of an angular velocity of motion of an aircraft
there appears at the output of the rate-of-turn indicator a signal,
which is proportional to this velocity, and which is amplified and
reaches the control-surface actuator. Based on the signal from the
RTI, the CSA deflects the control to the side counter to the rate
of turn. The operation of the damper in simplified form, without
taking the real characteristics of the amplifier and the control-
surface actuator into account, can be approximately described by
the following equation

$$\delta = k_\omega \cdot \omega \qquad (6.1)$$

From Formula (6.1) it follows that the use of oscillation
dampers is equivalent to the change in the derivative of damping of
the aircraft relative to the respective axis.

One basic disadvantage of a damper operating in accordance
with Equation (6.1) is that such a damper makes it more difficult
to establish steady banking at a constant angular velocity ω = const.
In fact with the motion of an aircraft at a certain constant angle
of velocity (to be specific let us look, for example, at the pitching

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damper), the damper which operates according to Diagram a will
accomplish the deflection of the rudder and inhibit development of
the angular yawing velocity. If this signal of the angular veloc-
ity is first transmitted to a high frequency filter (see Fig. 6.1,b)
which admits the variable but does not admit the constant signal
then during a steady maneuver when the angular velocity of the air-
craft is constant the signal from the damper to the control-surface
actuator will be equal to zero and the rudder is not deflected.
Figure 6.1,b shows the block schematic of a damper which in addition
to the elements noted above, includes a high frequency filter (HFF)
that serves for diminishing the disturbances from the damper during
the steady rolls.

As a high frequency filter an electronic circuit is usually
used, the operation of which can be described by the following
transmission function:

$$\left(\frac{x_{out}}{x_{inp}}\right) = \frac{T_1 p}{T_2 p + 1},$$ (6.2)

where p is the Laplace operator (operator of differentiation); T_1
and T_2 are constants of the filter.

Operation of the damper with a high-frequency filter may be
described by the following transmission function:

$$\delta = k_\omega \left(\frac{p}{T_2 p + 1}\right) \omega.$$ (6.3)

The difference in operation of the dampers of the two described
diagrams in practice appears only in analyzing steady systems of
motion. In analyzing the dynamics of an aircraft on a steady roll
the effect of the damper of the first type is equivalent to the
change in the derivative of the damper of the aircraft relative to
the respective axis and the effect of the second type damper does
not influence the motion of the aircraft and is not taken into
account. With oscillational motion of an aircraft the effect of
both types of dampers is approximately identical and equivalent to
the change in the derivative of damping of the aircraft relative to
the respective axis.

The effect of the pitching and yawing dampers on the dynamics
of the aircraft during rolling maneuvers has three basic types of
manifestation:

(1) Effect on the value of the free term $A_0(\overline{\omega}_x)$, i.e., on the
degree of stability of motion of the aircraft and the size of the
critical angular rolling velocities;

(2) Change in the static relationships and, respectively, the reaction of the aircraft to deflection of the controls;

(3) The effect on the degree of damping of oscillations during transient conditions.

The effect of the size of the derivatives of damping of the aircraft on the size of the free term $A_0(\bar{\omega}_x)$ of the characteristic equation of motion of the aircraft at a constant angular rolling velocity $\bar{\omega}_x$ = const was studied in Section 8, where it was shown that an increase in damping may lead to elimination of the zone of unstable motion of the aircraft and consequently to the disappearance of the critical rolling velocities. However, let us note that such /203 an effect on the functions $A_0(\bar{\omega}_x)$ is exerted only by the damper which operates according to the schematic shown on Figure 6.1,a. The dampers which operate according to the schematic shown on Figure 6.1,b have practically no influence on the steady motion of an aircraft and therefore exert no influence either on the value of $A_0(\bar{\omega}_x)$ or on the static solution. However, regardless of the fact that the proper choice of parameters of the damper in certain instances may succeed in eliminating the zone of unstable motion of an aircraft at a constant angular rolling velocity the effect of the dampers are far from being so simple and determinant. In a number of cases, the inclusion of a damper exerts an unfavorable influence on the dynamics of the spatial motion of an aircraft, in particular it will lead to a growth in values of the G-forces acting on it. Let us look separately at the influence of dampers of yawing and pitching on the dynamics of an aircraft during rolling maneuvers.

A. Dynamics of an Aircraft with Yawing Damper

The influence of a yawing damper, operating according to the plans shown on Figure 6.1,a, on the dynamics of an aircraft during rolling maneuvers is expressed basically in that because of the operation of the damper during maneuvers there occurs a decrease in the size of the angular yawing velocity ω_y. In this case as is easy to see from the equation of side slip, the decrease in ω_y when ω_x = const leads to a growth in the angle of side slip of the aircraft

$$\frac{d\beta}{d\tau} = \mu\bar{\omega}_y + \mu(\alpha_0 + \Delta\alpha)\bar{\omega}_x + \frac{c_z^\beta}{2}\beta. \tag{6.4}$$

This result can be made more precise if we use the formula for $A_{m\delta_e}^\beta$ from Table 2:

$$A_{m\delta_e}^\beta = \left(\frac{\beta_{ss}}{\Delta\bar{m}_z}\right) = \frac{\mu^2}{\Lambda_0}\left[B\frac{c_y^\alpha}{2} - \bar{m}_y^{\bar{\omega}_y}\right]. \tag{6.5}$$

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From Formula (6.5) it follows that the growth in the coefficient $\overline{m}_y^{\omega y}$ leads both to an increase in the denominator A_0/μ^2 and in the numerator of the expression. In this case the numerator with an increase in $\overline{m}_y^{\omega y}$ grows more rapidly as a result of which the change in the coefficient $\overline{m}_y^{\omega y}$ leads to an increase in the angle of side slip of the aircraft during the rolling maneuver. This result is quite important since it follows from it that the flight of the air- craft with the yawing damper turned on may lead to an increase in the lateral G-forces during the rolling maneuver. These results pertain also to the yawing damper operating according to this plan in Figure 6.1,a. However, the damper (see Fig. 6.1,b) increases only the value of β_{max} and does not influence the steady value of β_{ss}.
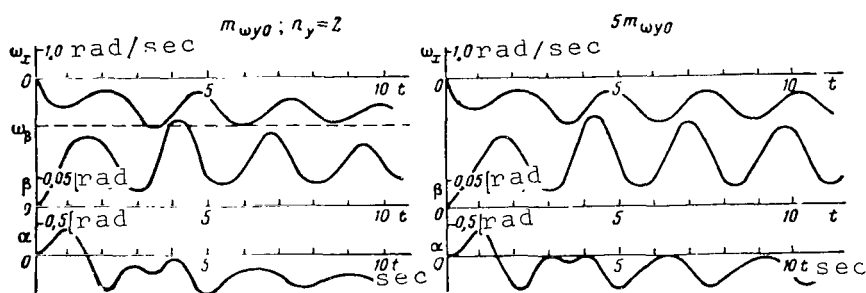

Fig. 6.2

Computations show that when $\omega_\beta \ll \omega_\alpha$ the yawing damper exerts a small influence on the transient conditions of an aircraft during a rolling maneuver (Fig. 6.2).

B. Dynamics of an Aircraft with a Pitching Damper

The effect of the pitching damper, operating according to the plan shown in Figure 6.1,a and b, on the motion of an aircraft during a rolling maneuver is on the whole analogous to the effect of the yawing damper.

Turning on the pitching damper leads to an increase in the reaction of the aircraft, for the angle of attack during a rolling maneuver, to the deflection of the rudder. This result follows in particular from the formula for the static derivative $A_{m\delta_r}^\alpha$ (see Table 2):

$$A_{m\delta_r}^\alpha = \left(\frac{\alpha_{ss}}{\Delta\overline{m}_y}\right) = \frac{\mu^2\overline{\omega}_x}{A_0}\left(\overline{m}_{zb}^{\overline{\omega}_z} + A\frac{c_z^\beta}{2}\right)$$

(6.6)

With a growth in the value $\overline{m}_{zb}^{\overline{\omega}z}$ the numerator in Expression (6.6) grows more rapidly than the denominator and the reaction of the

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aircraft, for the angle of attack α during the rolling maneuver, to the deflection of the rudder is amplified.

The pitching damper, operating both according to the plan in Figure 6.1,a and according to that in Figure 6.1,b, increases the damping of the yawing oscillations of the aircraft (when $\omega_\beta \ll \omega_\alpha$). Examples of the transient conditions are shown on Figure 6.3.

C.   Effect of the Rolling Damper

The effects of the rolling damper leads both to a change in the time of the transient conditions of the aircraft in rolling and to a decrease in the value of the steady angular rolling velocity <u>/205</u> (if the damper operates according to the plan shown on Figure 6.1,a).
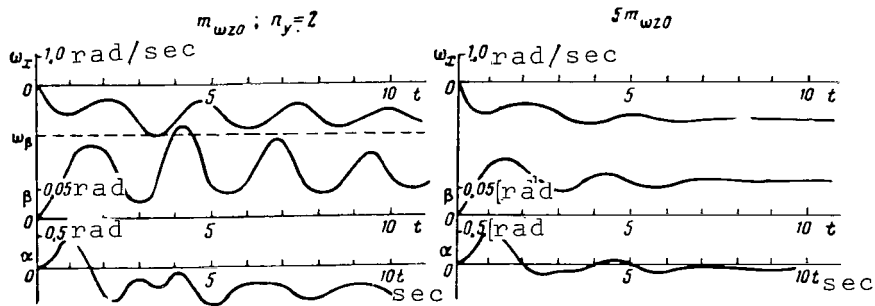

Fig. 6.3

Since the basic parameter in the analysis is the value of the angular rolling velocity, it is easy to see that turning the rolling damper on will lead only to a change in the quantitative relationships between the value of the aileron deflection and the reactions to it of the aircraft. If we also take into account that the value of the deflection of the ailerons from the rolling damper is usually quite small ($\Delta\delta_a = 2° - 3°$), we may then assume that its influence on the dynamics of the aircraft during the rolling maneuver is expressed mainly in the slowing of the reaction of the aircraft to the deflection of the ailerons which in certain instances may lower the G-forces acting on the aircraft. However, in summing up it follows to note that the use of dampers of rolling, yawing and pitching will not solve the problem of decreasing the G-forces during rolling maneuvers and does not exclude the possibility of it entering into a system of inertial rotation.

25.   Several Arguments for Choosing the Basic Parameters of Modern Aircraft

In studying the spatial motion of modern aircraft with strong rolling we must in first order find computations of the values of the maximal G-forces and the angles of attack and side slip during the maneuver and also analyze the possibility of loss by the air-

craft of controllability from the ailerons (the possibility of a
system of inertial rotation). The degree of importance of each of
these two questions, characteristic for the spatial maneuvers, is
determined by the flight system of the aircraft.

Let us look briefly at the influence of the systems of flight
and several parameters of the aircraft and its dynamics during
spatial maneuvers. Let us determine the computational ranges of
flight from the point of view of the maximal, normal and lateral
G-forces which arise during the rolling maneuver carried out from
the conditions of horizontal flight. We shall look at rolling
maneuvers which are carried out at a constant value of the aileron
deflection (δ_a = const).

Using the relationships given in Table 2 let us write the
formulas for $A_{\alpha_0}^{\alpha} ss$ and $A_{\alpha_0}^{\beta} ss$:

$$A_{\alpha_0}^{\alpha} ss = \left(\frac{\alpha_{ss}}{\alpha_0}\right) = \frac{\mu^2 \overline{\omega}_x^2}{A_0}\left(-\mu^2 \overline{\omega}_x^2 AB - \mu A \overline{m}_y^{\beta} - \overline{m}_y^{\overline{\omega}_y} \cdot \overline{m}_{z_b}^{\overline{\omega}_z}\right);$$ (6.7)

$$A_{\alpha_0}^{\beta} ss = \left(\frac{\beta_{ss}}{\alpha_0}\right) = \frac{\mu^2 \overline{\omega}_x}{A_0}\left(\overline{m}_{z_b}^{\alpha} \overline{m}_y^{\overline{\omega}_y} + \mu AB \overline{\omega}_x^2 \frac{c_y^{\alpha}}{2} + \frac{c_y^{\alpha}}{2\mu} \overline{m}_{z_b}^{\overline{\omega}_z} \overline{m}_y^{\overline{\omega}_y}\right)$$ (6.8)

If as α_0 we examine the angle of attack of the aircraft under
the conditions of horizontal flight ($\alpha_0 = \alpha_{hf}$), then these relation-
ships determine the mean values of the G-forces acting on the air-
craft

$$\left(\frac{\alpha_{ss}}{\alpha_0}\right) = \Delta n_y;$$

$$\left(\frac{\beta_{ss}}{\alpha_0}\right) = \Delta n_z \cdot \frac{c_y^{\alpha}}{c_z^{\beta}} .$$ (6.9)

If we ignore the influence of damping on the free term of the char-
acteristic equation A_0/μ^2 we can simplify Formulas (6.7) and (6.8).
Let us look at A_0/μ^2 in the form

$$\frac{A_0}{\mu^2} = \left(-\overline{m}_{z_b}^{\alpha} - A\mu \overline{\omega}_x^2\right)\left(-\overline{m}_y^{\beta} - B\mu \overline{\omega}_x^2\right).$$ (6.10)

Making the obvious transformations, we find

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$$\frac{a_{SS}}{a_0} \simeq \frac{1}{\left(\dfrac{\overline{\omega}_a}{\overline{\omega}_x}\right)^2 - 1} \; ; \tag{6.11}$$

$$\frac{\beta_{SS}}{a_0} \simeq \frac{\overline{m}_y^{\overline{\omega}_y} \cdot \overline{m}_z^a b}{B\mu\overline{\omega}_x \left[\left(\dfrac{\overline{\omega}_\beta}{\overline{\omega}_x}\right)^2 - 1\right]\left(-\overline{m}_z^a b - A\mu\overline{\omega}_x^2\right)} \; . \tag{6.12}$$

To find the value of $\overline{\omega}_x$ corresponding to a given constant deflection of the ailerons in all systems of flight let us look at the equations of moments relative to the axis OX_1:

$$\overline{m}_x^{\delta_a} \cdot \delta_a = -\left[\overline{m}_x^{\overline{\omega}} x + \overline{m}_x^\beta \frac{1}{\overline{\omega}_x}\left(\frac{\beta_{SS}}{a_0}\right)a_0\right]\overline{\omega}_x, \tag{6.13}$$

whence it follows that

$$\overline{\omega}_x = \frac{\ell - \overline{m}_x^{\delta_a} \cdot \delta_a}{\overline{m}_x^{\overline{\omega}}x + \overline{m}_x^\beta \frac{1}{\overline{\omega}_x}\left(\frac{\beta_{SS}}{a_0}\right)a_0} \; . \tag{6.14}$$

From Formula (6.14) it follows that the steady value of the angular /207 velocity ω_x is the greater as the angle of attack of the aircraft (a_0) is smaller under the conditions of horizontal flight.

Taking this result into account, from the analysis of Formula (6.11) we find that the normal G-force acting on the aircraft is the greater as the value of the critical rolling velocity is smaller (that is, the greater the height of flight) and the greater the angular rolling velocity $\overline{\omega}_x$ (the smaller a_0, i.e., the greater the reference pressure).
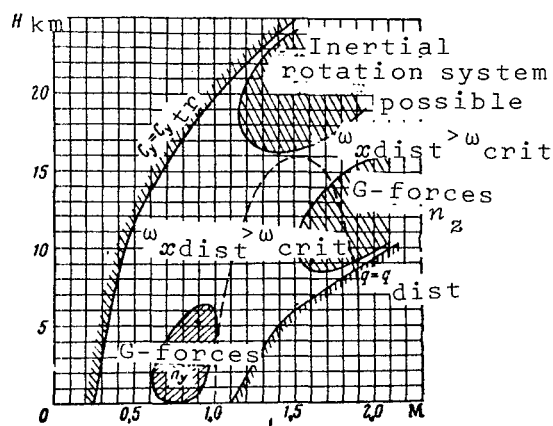
Since $\overline{\omega}_a$ is a small value when $M < 1$ and grows substantially when the number $M > 1$, from Formula (6.11) it follows that maximal normal G-forces may exist with rolling maneuvers carried out at subsonic flying speeds. The region in which large normal G-forces are possible is shown graphically on Figure 6.4.

From Formula (6.12) it follows that large lateral G-forces during

Fig. 6.4

rolling maneuvers may be developed both in cases when the critical velocity ω_α is the smaller and in the cases when the rolling velocity ω_β is the smaller. The computational case however is that when ω_β is the smaller critical rolling velocity, i.e., supersonic flying speeds. Since at supersonic speeds the relationships $\bar{\omega}_\alpha \gg \bar{\omega}_\beta$ are usually used, Formula (6.12) can then be simplified:

$$\frac{\beta_{ss}}{\alpha_0} \simeq \frac{-\bar{m}_y^{\bar{\omega}_y}}{B\mu\bar{\omega}_x\left[\left(\dfrac{\bar{\omega}_\beta}{\bar{\omega}_x}\right)^2 - 1\right]} \qquad (6.15)$$

The formula for (β_{ss}/α_0) has a form which is analogous to the formula (α_{ss}/α_0) studied above. The lateral G-forces are greater the smaller the value of the critical velocity ω_β and the larger the value of $\bar{\omega}_x$. From Relationship (6.14) it follows that ω_x grows with an increase in the reference pressure (with a decrease in α_0 = α_{hf}). The region in which the lateral G-force is the computed one is shown on Figure 6.4.

Finally let us evaluate the relative probabilities of the air- /208 craft entering into a system of inertial rotation. From the results given in the paper it follows that for modern aircraft of usual design, entry into a system of inertial rotation is possible at supersonic flying speeds (when $\bar{\omega}_\beta \ll \bar{\omega}_\alpha$) during rolling maneuvers carried out from conditions of flight with negative angles of attack.

For an aircraft to enter into the system it is necessary that the angular rolling velocity exceed the value of the first critical rolling velocity and the function $\Delta\bar{m}_{x_{ss}}(\bar{\omega}_x)$ have a zero for the angular rolling velocity greater than the second critical (maneuvers of C and E type, see Fig. 3.12). In such cases the greater the value of the lateral stability m_x^β, then the smaller are the deflections of the ailerons for possible entry of the aircraft into the system of inertial rotation.

Evaluation of the required deflections of ailerons can be carried out using the following formula:

$$\left|\bar{m}_x^\delta a \bar{\delta}_a\right| \geqslant \bar{m}_x^\omega x \bar{\omega}_\beta k_x \qquad (6.16)$$

or otherwise

$$\bar{\omega}_{x_{dist}} \geqslant \bar{\omega}_\beta k_x, \qquad (6.17)$$

where k_x is the empirical coefficient which takes the influence of m_x^β into account and $\alpha_0 < 0$.

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The greater the value of $|\overline{m}_x^\beta|$ the smaller is the value of the coefficient k_x.

From Expression (6.17) it follows that since $\overline{\omega}_\beta \sim \sqrt{-\overline{m}_\nu^\beta \varrho'}$, then the probability of entry into a system of inertial rotation grows with increase in the height of flight and with decrease in the excess directional stability. Increasing the excess lateral stability leads to a decrease in k_x which simplifies entry of the aircraft into large rolling velocities and increases the probability of its entering into a system of inertial rotation.

On the basis of the above discussions we can make the following conclusions:

1. Normal G-forces acting on an aircraft take maximal values for rolling maneuvers at subsonic flying speeds and the greater the value the smaller is the excess longitudinal stability of the aircraft. In this respect it is necessary that the excess longitudinal stability of the aircraft not be extraordinarily small.

2. Maximal lateral G-forces exist during rolling maneuvers carried out at supersonic flying speeds and are the greater in value as the excess directional stability of the aircraft is smaller.

3. The probability of the aircraft entering into a system of inertial rotation grows with decrease in the directional stability $(\overline{m}_y^\beta)$.

4. A large lateral stability makes it more difficult to pilot the aircraft and increases the probability that it will enter into a system of inertial rotation. Small values of the lateral stability of the aircraft $(\overline{m}_x^\beta)$ should then be guaranteed.

5. Carrying out rolling maneuvers with the dampers turned on somewhat facilitates piloting the aircraft since it slows down and "smooths" out its reaction to deflection of the controls.

Since in many cases the requirements formulated above for the aerodynamic parameters of stability of an aircraft cannot be satisfied completely, limitations are introduced for piloting the aircraft which are usually reduced to the following:

1. In a rolling maneuver with positive initial G-forces, the angles of bank cannot exceed 360°. With a large number of turns the angular rolling velocity must be limited.

2. In rolling maneuvers with initial G-force $n_y < 0.5$ the maximal angle of bank must not exceed 180° and escape from roll should begin at $\gamma \cong 90°$.

3. Any rolling maneuvers with negative initial G-force must be carried out with great care and should be avoided.

CHAPTER VII

MOTION OF SYMMETRIC ROTATING ROCKETS

The investigations carried out above and the qualitative re-
sults obtained confirm the great complexity of the problem of finding
solutions for the general case of motion of a solid, in this case
of an aircraft, under the influence of external moments linearly
depending on orientation and angular velocity of motion. In respect
to this, great interest is given to the study of more simple cases
of motion of an aircraft when several additional limitations are
applied to its aerodynamic and inertial characteristics. Of practi-
cal interest is the case for which such investigations can be car-
ried out more completely, that is the problem of motion of symmetric
controlled rockets or missiles rotating relative to the longitudinal
axis. This chapter is devoted to a study of the questions of the
dynamics of symmetric rotating missiles during motion with small
angles of attack and side slip. In this case as symmetric missiles
we will understand those for which the following conditions are
satisfied:

1. An ellipsoid of inertia of a missile is a body of rotation
the maximal axis of which corresponds with the longitudinal axis
OX_1.

2. The missile has at least two orthogonal planes of aerodynamic
symmetry passing through the longitudinal axis.

3. For a missile the aerodynamic moments are lacking which
act relative to the longitudinal axis and depend on the angles of
attack and side slip (moments of lateral stability or moments of
"transverse air-cooling").

Possible designs of these missiles are shown on Figure 7.1.

It is obvious that all results obtained above for the general
case can be expanded also to the specific case of motion of a sym-
metric missile, however there are several characteristics of such
motion. In first order we must note that the motion of statically
stable missiles in a rolling rotation are aperiodically stable at
any values of the angular rolling velocities (this result in particu-
lar was found in Section 8.). However under certain conditions we
may observe resonance phenomena which are near in essence to

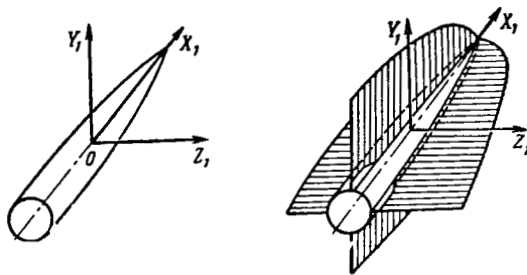aperiodic instability of an aircraft during critical angular rolling velocities (see Section 8). In general it follows to note that the angular velocities of rotation of missiles partially may be studied as constants and usually are significantly greater than the angular velocities of rotation characteristic of aircraft maneuvers. Therefore investigation of the motion of missiles rotating at a constant angular velocity have greater practical interest from the point of view of direct application of the results obtained than to analogous investigations for aircraft.

Fig. 7.1

26. Equations of Motion of a Symmetric Rotating Missile. Analysis of Stability

The equations of motion given in Chapter I can be expanded also to the motion of a symmetric missile, however in the latter case they must be additionally simplified. We should look at the motion of a missile under the same assumptions as the motion of an aircraft, i.e., we shall assume that the missile flies at a constant height, has a constant mass and the influence of gravity on its motion relative to the center of mass can be ignored. In this case the equations of motion of a missile written relative to the major inertial axis, lying in its pitching planes, in dimensional form (see Section 4) are of the type

$$
\alpha' - \mu\bar{\omega}_z + \mu^3\bar{\omega}_x = -\frac{c_y^\alpha}{2}\alpha;
$$

$$
\beta' - \mu\bar{\omega}_y - \mu\alpha\bar{\omega}_x = -\frac{c_z^\beta}{2}\beta;
$$

$$
\bar{\omega}_z' + A\mu\bar{\omega}_x\bar{\omega}_y = \bar{m}_z^\alpha\alpha + \bar{m}_z^{\omega_z}\bar{\omega}_z + \bar{m}_{0z};
$$

$$
\bar{\omega}_y' - B\mu\bar{\omega}_x\bar{\omega}_z = \bar{m}_y^\beta\beta + \bar{m}_y^{\omega_y}\bar{\omega}_y + \bar{m}_{0y};
$$

$$
\bar{\omega}_x' = \bar{m}_x^{\omega_x}\bar{\omega}_x + \bar{m}_{x0}.
$$

(7.1)

(7.2)

These equations describe the motion of a missile as a function of dimensionless time τ associated with the real time of the equation

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$$\tau = \int_0^t \frac{1}{\tau_m}\, dt, \qquad\qquad (7.3)$$

where

$$\tau_m = \frac{m}{\varrho s V(t)}.$$

Let us recall that in investigating the dynamics of missiles and the expressions for the derivatives, as the characteristic dimensions we usually take the area of the middle and the length of the body of the missile.

Let us make the transformations of Equations (7.1) and (7.2) to a form which is convenient for analysis. On the strength of the aerodynamic and inertial symmetry of the missile we can introduce the following definitions:

$$c_y^{\alpha} = -c_z^{\beta} = c_{\varphi}; \quad \overline{m}_z^{\alpha} = \overline{m}_y^{\beta} = m_{\tau}; \quad \overline{m}_z^{\overline{\omega}_z} = \overline{m}_y^{\overline{\omega}_y} = m_{\omega};$$

$$A = B = 1 - \overline{I}_x,$$

where

$$\overline{I}_x = \frac{I_x}{I}.$$

The terms $\overline{m}_{0x}$, $\overline{m}_{0y}$, and $\overline{m}_{0z}$ which enter into the right-hand side of Equations (7.1) and (7.2) represent moments of small non-symmetry which may arise either due to the aerodynamic non-symmetry (small inequality of the angles of setting up the carrier surfaces, small aerodynamic non-symmetry of shape, etc.) or with respect to the action of constant disturbed moments of the thrust misalignment of the engine on the active section, etc.

From Equations (7.1) and (7.2) it is obvious that the rolling motion of a missile, if the influence of the moments of the transverse air-cooling $\overline{m}_x(\alpha, \beta)$ can be ignored (for example, for missiles near the axisymmetric), does not depend on the pitching and yawing motion of the missile and when $\overline{m}_{x0} = 0$ the angular rolling velocity $\overline{\omega}_x$ is constant in value. Hence in particular it follows that Equations (7.1) may be studied separately from Equations (7.2) and we assume that $\overline{\omega}_x = $ const.

The symmetric form of the equations of motion (7.1) permit simplifying their writing somewhat due to the transition to complex variables, i.e., a complex angle of attack and angular velocity using the relationship

$$\varphi = \alpha + i\beta; \\ \omega = \overline{\omega}_z + i\overline{\omega}_y. \Bigg\} \qquad (7.4)$$

Making a simple transformation we find

$$\varphi' - \mu\omega - i\mu\Omega\varphi = -\frac{c_\varphi}{2}\varphi; \\ \omega' - iA\mu\Omega\omega = m_\varphi\varphi + m_\omega\omega + m_0, \Bigg\} \qquad (7.5)$$

where

$$\Omega = \overline{\omega}_x;$$

$$m_0 = \overline{m}_{0z} + i\overline{m}_{0y}. \qquad (7.6)$$

The system of Equations (7.5) can be conveniently reduced to a single equation which describes the change in the complex angle of attack of the missile ϕ. Let us make these transformations for motion at a constant angular velocity $\overline{\omega}_x$. From the first equation of system (7.5) we find

$$\omega' = \frac{1}{\mu}\left[\varphi'' + \left(\frac{c_\varphi}{2} - i\mu\Omega\right)\varphi'\right]; \qquad (7.7)$$

$$\omega = \frac{1}{\mu}\left[\varphi' + \left(\frac{c_\varphi}{2} - i\mu\Omega\right)\varphi\right]. \qquad (7.8)$$

Substituting Expressions (7.7) and (7.8) into the second equation of system (7.5) and grouping the terms for the respective derivatives of the angle ϕ, we find

$$\varphi'' + \left[\frac{c_\varphi}{2} - m_\omega - i\mu\Omega(1 + A)\right]\varphi' + \left[-\mu m_\varphi - \mu^2 A\Omega^2 - \right.$$

$$\left. - \frac{m_\omega c_\varphi}{2} - i\mu\Omega\left(A\frac{c_\varphi}{2} - m_\omega\right)\right]\varphi = m_0\mu. \qquad (7.9)$$

Equation (7.9) is a linear nonuniform equation with constant complex coefficients. For its solution we must use the methods of solving linear equations with real coefficients. The basic difference consists of the fact that the arbitrary constants in solving the equation with complex coefficients are complex numbers and the roots of the characteristic equation cannot be complex-conjugate as is the case with the equation having real coefficients. Taking into account the notations made, the solution to Equation (7.9) can

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be written in the form

$$\varphi = A_1 e^{\lambda_1 \tau} + A_2 e^{\lambda_2 \tau} + A_3, \tag{7.10}$$

where the coefficients A_1, A_2 and A_3 are complex constants depending on the initial conditions. The first two terms of Expression (7.10) describe the general solution to the uniform Equation (7.9) and the /214 coefficient A_3 is the specific solution to the nonuniform equation

$$A_3 = \cfrac{m_0 \mu}{-\mu m_\varphi - \mu^2 A \Omega^2 - \dfrac{m_\omega c_\varphi}{2} - i\mu\Omega \overline{\left(A\dfrac{c_\varphi}{2} - m_\omega \right)}}. \tag{7.11}$$

The roots λ_1 and λ_2 are determined from the expression

$$\lambda_{1,2} = \frac{\left[\dfrac{c_\varphi}{2} - m_\omega - i\mu\Omega(1+A) \right]}{2} \pm$$

$$\pm \frac{1}{2} \sqrt{\left[\frac{c_\varphi}{2} - m_\omega - i\mu\Omega(1+A) \right]^2 - } \tag{7.12}$$

$$\sqrt{\overline{-4 \left\{ -\mu m_\varphi - \mu^2 A\Omega^2 - \frac{c_\varphi m_\omega}{2} - i\mu\Omega \left(A\frac{c_\varphi}{2} - m_\omega \right) \right\}}}.$$

From Expression (7.12) it is obvious that since the coefficients of control of motion (7.9) are complex, then the roots $\lambda_{1,2}$ cannot be complex-conjugates and in the general case are different complex roots and the solution (7.10) consists of oscillations of two different frequencies. Taking into account that the square root in Expression (7.12) represents a complex number the expression for the roots of the characteristic equation can be rewritten in the form

$$\lambda_{1,2} = -(\text{Re}_1 \pm \text{Re}_2) + i(\text{Im}_1 \pm \text{Im}_2), \tag{7.13}$$

where $(-\text{Re}_1, \text{Im}_1)$ and $(-\text{Re}_2, \text{Im}_2)$ are real and imaginary parts, respectively, of the first and second terms in Relationship (7.12).

For stability of motion we must satisfy the conditions

$$\text{Re}_1 > 0; \tag{7.14}$$

$$\text{Re}_1 > |\text{Re}_2|. \tag{7.15}$$

Let us look at an analysis of these inequalities. Taking into account that

$$\text{Re}_1 = \left(\frac{c_\varphi}{2} - m\omega\right)\frac{1}{2},$$

we find that Condition (7.14) is always satisfied, therefore we must analyze only Conditions (7.15). Taking into account that the damping is small, on the basis of Relationship (7.15), we introduce the approximate condition of stability. In the subroot expression of Formula (7.12) in the case of small damping the coefficient in the imaginary part is substantially less in value than in the real part, therefore in extracting the root we can retain only the first two terms of its expansion into a Taylor series. Carrying out this /215 operation we find an approximate expression for the roots λ_1 and λ_2.

$$\lambda_{1,2} = -\frac{\left[\dfrac{c_\varphi}{2} - m_\omega - l\mu\Omega(1+A)\right]}{2} \pm \frac{1}{2}\left\{-\frac{\mu\Omega(1-A)\left(\dfrac{c_\varphi}{2} + m_\omega\right)}{\sqrt{-4\mu m_\varphi + \mu^2\Omega^2(1-A)^2}} + \right.$$
$$\left. + i\sqrt{-4\mu m_\varphi + \mu^2\Omega^2(1-A)^2}\right\}. \qquad (7.16)$$

Using Expression (7.16) the condition of stability (7.15) can be written in the form

$$\frac{m_\varphi}{\mu\Omega^2} + \frac{1}{4}(\overline{I}_x)^2 \geqslant \frac{(\overline{I}_x)^2\left(\dfrac{c_\varphi}{2} + m_\omega\right)^2}{4\left(\dfrac{c_\varphi}{2} - m_\omega\right)^2}. \qquad (7.17)$$

Inequality (7.17) has been obtained on the assumption that the expression under the radical in Formula (7.16) is positive. In other words we additionally assume that the following inequality is satisfied

$$m_\varphi \leqslant \frac{\mu\Omega^2(\overline{I}_x)^2}{4}. \qquad (7.18)$$

Inequality (7.18) is an approximate condition of the aperiodic stability of motion of a missile. From comparison of conditions (7.17) and (7.18) it is clear that the condition of stability (7.17) is stronger and its satisfaction certainly involves

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satisfying Conditions (7.18). In fact, Condition (7.17) can be rewritten in the following form:

$$m_\varphi \leqslant \frac{\mu \Omega^2 (\bar{I}_x)^2}{4} K,$$

(7.19)

where

$$K = 1 - \left(\frac{\dfrac{c_\varphi}{2} + m_\omega}{\dfrac{c_\varphi}{2} - m_\omega} \right)^2.$$

(7.20)

The value of K, as follows from Expression (7.20), lies in the range between zero and unity:

$$0 < K < 1.$$

(7.21)

When $K = 1$ we find the approximate conditions of the aperiodic stability (7.18); on the other hand if $K = 0$ then only the statically stable missile may be the dynamically stable one. In practice the value of K is neither zero nor one, but occupies certain mean values.

In estimating the aperiodic stability of uncontrolled rotating missiles we usually use Conditions (7.18) with the help of which we can select the required value of the angular velocity of rotation of the missile which will guarantee stability. If in this case Condition (7.17) is not satisfied then the missile is found to be oscillationally unstable, however these oscillations diverge slowly since the values c_ϕ and m_ω are usually small.

27. Dynamics of a Symmetric Rotating Missile. Resonance.

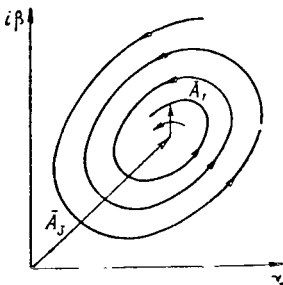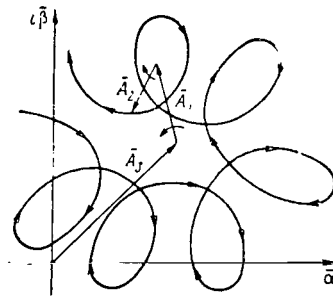In Section 26 we determined the basic conditions of stability

Fig. 7.2

Fig. 7.3

of motion of a missile and found the general solution for the complex angle of attack $\phi(\tau)$. As follows from the writing of the solution itself, $\phi(\tau)$ may be studied as the sum of three vectors on the plane α, $i\beta$: two vectors describe the oscillational motion of a missile and the third vector describes the shift of the zero deflection caused by the constant external moment of nonsymmetry. Examples of motion of a missile on the plane α, $i\beta$ are shown on Figures 7.2 and 7.3 (let us recall that the complex angle $\phi(\tau)$ is determined in body axes of the missile). On Figure 7.2 the motions of a non-rotating missile is plotted. In this case in the solution there is a vector of constant shift $\overline{A}_3$ caused by the effect of aerodynamic nonsymmetry and the vector $\overline{A}_1$ corresponding to the oscillational moment of the missile. Change in the angle of attack and side slip of the missile occurs at an identical frequency and therefore in the case of zero damping is represented in the form of an ellipse. An example of the motion of a rotating missile is shown on Figure 7.3. In this case the motion is described by three vectors, the fixed vector $\overline{A}_3$ and two vectors $\overline{A}_1$ and $\overline{A}_2$ which correspond to the motion with two frequencies of oscillation of the missile. Vector $\overline{A}_1$ describes the high-frequency, and vector $\overline{A}_2$ the low-frequency, components of the solution.

The dynamics of a missile can be studied more conveniently and clearly if we take as the system of coordinate axes the axes $OX*Y*Z*$, which oscillate along with the missile according to α and β but do not rotate. The relationship between the systems of co-ordinate axes is clear from Figure 7.4. The nonrotating system of coordinates $OX*Y*Z*$ is convenient in that it permits tracing the trajectory of motion of the nose of the missile around the velocity vector and in the same way allows the picture of its motion to be represented more clearly.
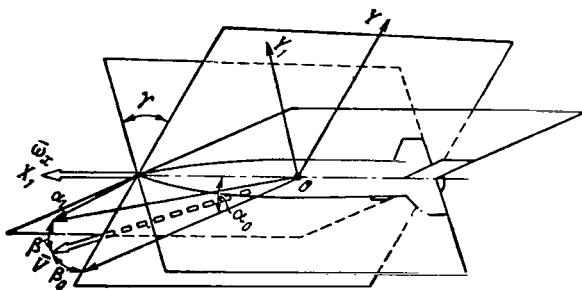

Fig. 7.4

To derive equations of motion of a missile in the axes $OX*Y*Z*$ we can use Equation (7.9) and convert

$$\overline{\varphi} = \varphi e^{-i\Omega\mu\tau}, \qquad (7.22)$$

which places the angle ϕ in agreement with the rotating system of coordinates, and the angle $\overline{\phi}$ into a non-rotating system. Substituting into Equation (7.9) relationship (7.22) we find the equation for $\overline{\phi}$:

$$\overline{\varphi}'' + \left[\frac{c_\varphi}{2} - m_\omega - i\mu\Omega\overline{I}_x\right]\overline{\varphi}' + \qquad (7.23)$$

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$$+\left[-\bar{\mu}m_\varphi-\mu^2\Omega^2\overline{J}_x-i\mu\Omega\left(A\,\frac{c_\varphi}{2}-m_\omega-1\right)\right]\bar{\varphi}=$$

$$=m_0\mu e^{-i\Omega\mu\tau}.$$

(7.23)

The solution for ϕ can be more simply obtained by substituting the variable (7.22) in solution (7.10) for the angle ϕ:

$$\bar{\varphi}=A_1e^{(\lambda_1-i\Omega\mu)\tau}+A_2e^{(\lambda_2-i\Omega\mu)\tau}+A_3e^{-i\Omega\mu\tau}.$$

(7.24)

If we introduce new definitions for the roots of the characteristic <u>/218</u> equations we find

$$\bar{\varphi}=A_1e^{\bar{\lambda}_1\tau}+A_2e^{\bar{\lambda}_2\tau}+A_3e^{-i\Omega\mu\tau},$$

(7.25)

where

$$\bar{\lambda}_1=-\frac{1}{2}\left[\left(\frac{c_\varphi}{2}-m_\omega\right)+\frac{\overline{J}_x\left(\frac{c_\varphi}{2}+m_\omega\right)}{2\sqrt{-\frac{m_\varphi}{\mu\Omega^2}+\left(\frac{\overline{J}_x}{2}\right)^2}}\right]-$$
$$-i\left[\mu\Omega\,\frac{\overline{J}_x}{2}-\mu\Omega\sqrt{-\frac{m_\varphi}{\mu\Omega^2}+\left(\frac{\overline{J}_x}{2}\right)^2}\right];$$

(7.26)

$$\bar{\lambda}_2=-\frac{1}{2}\left[\left(\frac{c_\varphi}{2}-m_\omega\right)-\frac{\overline{J}_x\left(\frac{c_\varphi}{2}+m_\omega\right)}{2\sqrt{-\frac{m_\varphi}{\mu\Omega^2}+\left(\frac{\overline{J}_x}{2}\right)^2}}\right]-$$
$$-i\left[\mu\Omega\frac{\overline{J}_x}{2}+\mu\Omega\sqrt{\left(-\frac{m_\varphi}{\mu\Omega^2}+\left(\frac{\overline{J}_x}{2}\right)^2\right)}\right].$$

(7.27)

From Expressions (7.26) and (7.27) it is clear that for a statically unstable missile ($m_\phi > 0$) the imaginary parts of the roots $\bar{\lambda}_1$ and $\bar{\lambda}_2$ have <u>identical</u> signs, i.e., the respective vectors on the complex plane $\bar{\alpha}$, $\bar{i\beta}$ rotate to one <u>side</u> with an increase in τ. In this case the motion on the plane ($\bar{\alpha}$, $\bar{i\beta}$) for a symmetric missile when $A_3 \equiv 0$ has the form shown on Figure 7.5. If the missile is <u>statically</u> stable ($m_\phi < 0$) then the imaginary parts of the roots <u>/219</u> $\bar{\lambda}_1$ and $\bar{\lambda}_2$ have <u>different</u> signs and the respective vectors on the complex plane ($\bar{\alpha}$, $\bar{i\beta}$) rotate to the opposite sides. As a result the motion for a symmetric missile has the form shown on Figure 7.6. From Expressions (7.26) and (7.27) we can find the character of the

dependence of the roots on the value of the angular rolling veloc-
ity both for the stable and the statically unstable missile. The
character, more precisely the tendency, of the change of the roots
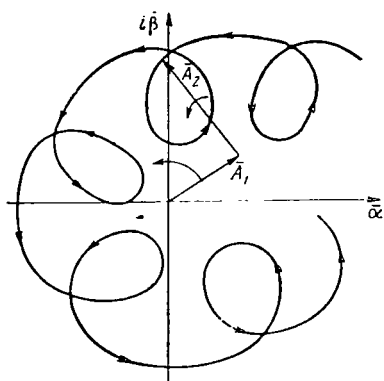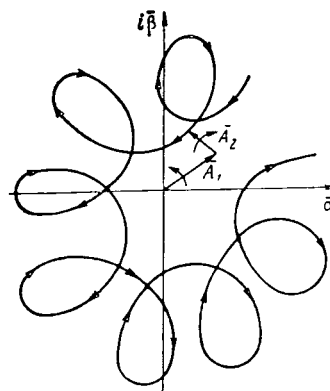in increasing the angular rolling velocity is illustrated on Tables
7 and 8.

Fig. 7.5   Fig. 7.6

TABLE 7. STATICALLY STABLE MISSILE

Roots Parameters of the missile	$\|\mathrm{Re}\,\overline{\lambda}_1\|$	$\|\mathrm{Im}\,\overline{\lambda}_1\|$	$\|\mathrm{Re}\,\vec{\lambda}_2\|$	$\|\mathrm{Im}\,\overline{\lambda}_2\|$
$\dfrac{c_\varphi}{2} > -m_\omega$	Grows	Grows	Drops	Grows
$\dfrac{c_\varphi}{2} < -m_\omega$	Drops	Grows	Grows	Grows

TABLE 8. STATICALLY UNSTABLE MISSILE

Parameters of the missile \ Roots	$\lvert \mathrm{Re}\,\overline{\lambda}_1 \rvert$	$\lvert \mathrm{Im}\,\overline{\lambda}_1 \rvert$	$\lvert \mathrm{Re}\,\overline{\lambda}_2 \rvert$	$\lvert \mathrm{Im}\,\overline{\lambda}_2 \rvert$
$\dfrac{c_\varphi}{2} > -m_\omega$	Drops	First grows, then drops	Grows	Grows
$\dfrac{c_\varphi}{2} < -m_0$	Grows	Grows, then drops	Drops	Grows

From Tables 7 and 8 it is clear that the rules in the changes /220 of roots as a function of the angular rolling velocity Ω substantially depend on the sign of the coefficient of the static stability of the missile. In particular the direction of the change of the real parts of the roots and consequently the degree of damping of the motion have a directly opposite character for the statically stable and unstable missile. Let us proceed to a more detailed analysis of the specific solution to a nonuniform equation obtained due to the presence of a constant disturbing moment of aerodynamic nonsymmetry of the aircraft. The expression for A_3, as was found earlier, has the form

$$A_3 = \frac{m_0}{-\left(m_\varphi + \dfrac{c_\varphi m_\omega}{2\mu}\right) - \mu A \Omega^2 - i\Omega\left(A\dfrac{c_\varphi}{2} - m_\omega\right)}. \qquad (7.28)$$

The absolute value of A_3 characterizes the amplitude of the missile from the zero position caused by the constant disturbing moment. The absolute value of A_3 assumes a maximal value when the absolute value of the denominator is minimal. The square of the absolute value of the complex function standing in the numerator of (7.28) is determined from the formula

$$\Phi(\Omega^2) = \left(m_\varphi + \frac{c_\varphi m_\omega}{2\mu} + \mu A \Omega^2\right)^2 + \Omega^2\left(A\frac{c_\varphi}{2} - m_\omega\right)^2 \qquad (7.29)$$

It is easy to prove that the function $\Phi(\Omega^2)$ agrees with the free term of the characteristic equation A_0 obtained in Section 8 in which the parameters of the symmetric missile are given. Expression (7.28) represents simply the static solution written for the complex angle of attack from the external disturbing moment

analogous to the relationships given for the general case of a nonsymmetric aircraft in Table 2.

From Expression (7.29), by equating the derivative $\partial\Phi/\partial\Omega^2$ to zero, it is easy to find an expression for the angular rolling velocity Ω for which $\Phi(\Omega^2)$ has a minimal value. Designating this expression by Ω_{crit}, we obtain

$$- \Omega_{crit} \cong \sqrt{-\frac{m_\varphi}{\mu A} - \frac{\left(A\dfrac{c_\varphi}{2} - m_\omega\right)}{2\mu^2 A^2}} . \qquad (7.30)$$

For this value of the angular rolling velocity ($\Omega = \Omega_{crit}$), the absolute value of A_3 assumes a maximal value equal to

$$\max|A_3| = \frac{|m_0|}{\left(A\dfrac{c_\varphi}{2} - m_\omega\right)\sqrt{-\dfrac{m_\varphi}{\mu A} - \dfrac{A\dfrac{c_\varphi}{2} - m_\omega}{2\mu^2 A^2}}} . \qquad (7.31)$$
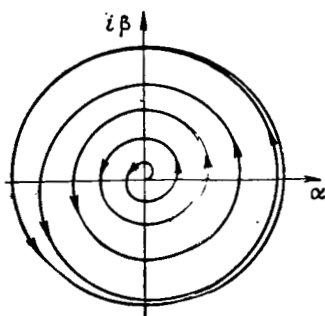
Thus, from the analysis given it follows that with a certain /221 value of the angular rolling velocity equal to Ω_{crit} the specific solution to the nonuniform equation grows substantially. In particular, with the disappearance of small damping, it tends to infinitely large value. The physical sense of the result obtained becomes more clear if we return to Equation (7.23). The specific solution to Equation (7.23) is found as a result of the effect on the missile of the periodic disturbance of the angular velocity $\mu\Omega$. When the periodic disturbance acts at a frequency equal to the natural frequency of the oscillations of the system the phenomenon of resonance begins. Direct proof is easy to find that the angular

Fig. 7.7

velocity Ω, corresponding to the resonance, is determined using Formula (7.30). Thus for symmetric missiles the existence of a small nonsymmetry leads to the appearance of a new effect in their dynamics, that is, to a resonance growth in the angles of attack and side slip. In connection with this the stable missile in the presence of small nonsymmetry and with motion at an angular rolling velocity equal to the critical may be found to be practically unstable (Fig. 7.7). Direct comparison of Expressions (7.29) and (2.4) indicates a strong relationship between the resonance phenomena characteristic for the symmetric missiles and the more complex

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of the dynamics of nonsymmetric aircraft.

For small values of damping, Formula (7.30) can be simplified

$$\Omega_{crit} \cong \sqrt{-\frac{m_\varphi}{\mu A}}. \tag{7.32}$$

It should be noted that the resonance caused by the rolling rotation and the lateral oscillation, as follows from Expression (7.30), is possible only for statically stable missiles and impossible for statically unstable missiles. In all cases of motion of a statically unstable missile, for example an artillary shell, the rolling rotation stabilizes its motion [61b].

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